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One-Dimensional Harmonic Lattice Caricature of Hydrodynamics

R. L. Dobrushin,[1] A. Pellegrinotti,[2] Yu. M. Suhov,[1] and L. Triolo[3]

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We derive the hydrodynamic (Euler) approximation for the harmonic time evolution of infinite classical oscillator system on one-dimensional lattice $\mathbb{Z}^1$. It is known that equilibrium (i.e., time-invariant attractive) states for this model are translationally invariant Gaussian ones, with the mean 0, which satisfy some linear relations involving the interaction quadratic form. The natural "parameter" characterizing equilibrium states is the spectral density matrix function (SDMF) $\hat{F}(\theta)$, $\theta \in [-\pi, \pi)$. Time evolution of a space "profile" of local equilibrium parameters is described by a space-time SDMF $\hat{F}(t; x, \theta)$ t, $x \in \mathbf{R}^1$. The hydrodynamic equation for $\hat{F}(t; x, \theta)$ which we derive in this paper means that the "normal mode" profiles indexed by θ are moving according to linear laws and are mutually independent. The procedure of deriving the hydrodynamic equation is the following: We fix an initial SDMF profile $\hat{F}(x, \theta)$ and a family $\{P^\varepsilon, \varepsilon > 0\}$ of mean 0 states which satisfy the two conditions imposed on the covariance of spins at various lattice points: (a) the covariance at points "close" to the value $\varepsilon^{-1}x$ in the state P^ε is approximately described by the SDMF $\hat{F}(x, \theta)$; (b) The covariance (on large distances) decreases with distance quickly enough and uniformly in ε. Given nonzero $t \in \mathbb{R}^1$, we consider the states $P^\varepsilon_{\varepsilon^{-1}t}$, $\varepsilon > 0$, describing the system at the time moments $\varepsilon^{-1}t$ during its harmonic time evolution. We check that the covariance at lattice points close to $\varepsilon^{-1}x$ in the state $P^\varepsilon_{\varepsilon^{-1}t}$ is approximately described by a SDMF $\hat{F}(t; x, \theta)$ and establish the connection between $\hat{F}(t; x, \theta)$ and $\hat{F}(x, \theta)$.

KEY WORDS: Harmonic oscillators; Gibbs states and conserved quantities; hydrodynamic limit.

[1] Institute for Problems of Information Transmission, USSR Academy of Sciences, 101447 GSP-4, Moscow, USSR.

[2] Dipartimento di Matematica e Fisica, Universita di Camerino, 62032 Camerino, Italy and GNFM-CNR.

[3] Dipartimento di Metodi e Modelli Matematici, Universita di Roma "La Sapienza," 00100 Roma, Italy and GNFM-CNR.

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1. INTRODUCTION

There has been in recent years intensive development and rigorous investigation of the problem of derivation of the kinetic (Boltzmann, Vlasov, Landau) and hydrodynamic (Euler, Navier–Stokes) equations from the Hamilton dynamics in a large (or infinite) particle system. The statement and first stages in the investigation of this fundamental problem were completed by Bogoliubov, Grad, Morrey, and Uhlenbeck et al. in the period from 1946 to 1955. The present state of the problem, including new results and methods on this topic, is discussed in review papers[1,2,3] to which we refer the reader for the details and explanations of the approach we adopt in this work.

The "modern" point of view on the problem under consideration is that the above-mentioned equations may be naturally obtained in the course of special limit procedures involving some "scaling" of space, time, and interaction. In particular, notions of "macroscopic" and "microscopic" space- and time-variables are naturally arising; their ratio characterizes the inhomogeneity of the system and tends to infinity in the course of the corresponding limit.

Therefore, kinetic and hydrodynamic equations should be regarded as specific approximations for the time evolution generated by a Hamilton dynamic. For instance, the Boltzmann equation may be obtained after so-called low density (Boltzmann–Grad) limit,[4] the Vlasov equation after mean field limit,[5,6] and so on.

Hydrodynamic equations which are expected to be obtained in the so-called hydrodynamic limit procedure are of particular interest from mathematical and applicative points of view. These equations describe time evolution of a macroscopic space distribution (space profile) of "local equilibrium" parameters which are, in general, given by the three conserved quantities: the fluid mass density, velocity momentum, and internal energy (temperature) at (macroscopic) space point x. Therefore, the rigorous foundation of hydrodynamic approximation requires the proof of the fact that the Hamilton time evolution brings the state of the system "locally close" to equilibrium ones (by "state" we mean in this paper a probability measure on the phase space of the system under consideration). In particular, this requires the proof of convergence to an equilibrium state for a wide class of initial states. However, such a problem seems to be, in general, beyond the reach of available mathematical tools. Hence, it is interesting to study "exact solvable" models of time evolution for which the convergence to an equilibrium state may be proven for large classes of initial states.

The simplest model of such type (which leads to a trivial hydrodynamics) is the free classical gas. However, this model is degenerated: the set of equilibrium states (i.e., time invariant states which are limiting ones for large classes of initial states) is much "richer" in this case than in the general case. It is characterized by a "functional" parameter instead of the finite set of the above-mentioned parameters.

Similar features are peculiar for models which are connected with the free gas. One of such models, the one-dimensional hard rod model, has been studied.[7] The degeneracy of this model leads to an "exotic" hydrodynamics; this explains the term "caricature" used therein.

This paper is devoted to another degenerate model, that of classical harmonic oscillators. This model, like the preceding degenerate ones, exhibits a "nonstandard" hydrodynamics: it continues the list of caricature models initiated in Ref. 7. As in Ref. 7, the authors think that the study of the "exotic" hydrodynamics for degenerate systems is useful for a better understanding of the "normal" hydrodynamics as limit behavior of realistic systems.

We consider the system of classical "spins" with the single phase space $\mathbb{R}^1 \times \mathbb{R}^1$ which are indexed by points of one-dimensional lattice $\mathbb{Z}^1$. The formal hamiltonian of the model is given by

$$\frac{1}{2} \sum_{j \in \mathbb{Z}^1} p_j^2 + \sum_{j, j' \in \mathbb{Z}^1} V(j'-j)\, q_j q_{j'}$$

q_j being the position (displacement) and p_j the momentum for the spin at the point $j \in \mathbb{Z}^1$. The interaction between spins in our model depends on their displacements and is described by a translationally invariant quadratic form

$$\sum_{j, j' \in \mathbb{Z}^1} V(j-j')\, q_j q_{j'}$$

We denote by $\omega^2(\theta)$, $\theta \in [-\pi, \pi)$, the Fourier transform

$$\omega^2(\theta) = \sum_{k \in \mathbb{Z}^1} \exp(ik\theta)\, V(k)$$

this is a real even function on $[-\pi, \pi)$, as $V(j'-j) = V(j-j')$.

Convergence to equilibrium states for this model has been proven in Ref. 8: This is the starting point for our investigation. The limit states are translationally invariant Gaussian ones, with mean zero, which satisfy an "equilibrium condition": a linear relation involving the interaction quadratic form. This relation may be expressed in terms of the covariance of the displacement-momentum vectors $y_j = (q_j, p_j)$ for spins at various points $j, j' \in \mathbb{Z}^1$

$$F_{j-j'} = \begin{pmatrix} \langle q_j q_{j'} \rangle & \langle q_j p_{j'} \rangle \\ \langle p_j q_{j'} \rangle & \langle p_j p_{j'} \rangle \end{pmatrix} \tag{1.1}$$

From the technical point of view it is more convenient to deal with the Fourier transform of the covariance, which is a complex (2×2) matrix function on $[-\pi, \pi)$, and call spectral density matrix function (SDMF)

$$\hat{F}(\theta) = \sum_{k \in \mathbb{Z}^1} F_k \exp(ik\theta) \tag{1.2}$$

By definition, the SDMF of an arbitrary translationally invariant state should satisfy a number of conditions. Namely, the diagonals $\hat{F}^{1,1}$ and $\hat{F}^{2,2}$, corresponding to $\langle q_j q_{j'} \rangle$ and $\langle p_j p_{j'} \rangle$, are nonnegative even functions on $[-\pi, \pi)$, and the off-diagonals $\hat{F}^{1,2}$ and $\hat{F}^{2,1}$, corresponding to $\langle q_j p_{j'} \rangle$ and $\langle p_j q_{j'} \rangle$, obey

$$\hat{F}^{1,2}(-\theta) = \overline{\hat{F}^{1,2}(\theta)} = \overline{\hat{F}^{2,1}(-\theta)} = \hat{F}^{2,1}(\theta), \qquad \theta \in [-\pi, \pi]$$

Furthermore, for any θ, the matrix $\hat{F}(\theta)$ is positively semidefinite.

The equilibrium condition characterizes time-invariant SDMFs. They must be of the form

$$\hat{F}(\theta) = \begin{pmatrix} \hat{g}(\theta) & \hat{h}(\theta) \\ -\hat{h}(\theta) & \omega(\theta)^2 \hat{g}(\theta) \end{pmatrix} \tag{1.3}$$

where $\hat{g}$ is a nonnegative even and $\hat{h}$ an odd, purely imaginary, function on $[-\pi, \pi)$.

Thus, according to a general point of view, the hydrodynamic equations in this model will describe the time evolution of a family of SDMFs depending on macroscopic space point $x \in \mathbb{R}^1$ (macroscopic space SDMF profile). In this paper we consider, as in [7], only the Euler equation (zero-order approximation). The hydrodynamic limit procedure is performed in the following way. Given an initial macroscopic space SDMF profile $\{\hat{F}(x, \cdot),\ x \in \mathbb{R}^1\}$ we consider a family of initial states P^ε, $\varepsilon > 0$, which satisfies the following conditions: (a) for any $x \in \mathbb{R}^1$ the covariance in the state P^ε at (microscopic) points $j, j' \in \mathbb{Z}^1$, which differ from $\varepsilon^{-1}x$ by $o(\varepsilon^{-1})$, is approximately described (as $\varepsilon \to 0$) by the SDMF $\hat{F}(x, \cdot)$; (b) the covariance in the state P^ε at (microscopic) points $j, j' \in \mathbb{Z}^1$ vanishes when $|j - j'| \to \infty$ uniformly in ε and quickly enough.

Condition (a) means that the macroscopic inhomogeneity of the states P^ε, $\varepsilon > 0$ (in the sense of the covariance) is described in the limit $\varepsilon \to 0$ by the initial SDMF profile $\{\hat{F}(x, \cdot)\}$. Condition (b) is a sort of so-called "chaos" hypothesis; it leads to an asymptotical total independence of macroevents in different macroscopic space points.

Now we "switch-in" the harmonic time evolution (with a fixed interaction not depending on ε). Given nonzero $t \in \mathbb{R}^1$ (a macroscopic time moment), we regard the states $P^{\varepsilon}_{\varepsilon^{-1}t}$, $\varepsilon > 0$, which describe the system at the microscopic time $\varepsilon^{-1}t$. Our aim is to verify that for any $x \in \mathbb{R}^1$ the covariance in the state $P^{\varepsilon}_{\varepsilon^{-1}t}$ at points $j, j' \in \mathbb{Z}^1$, which differ from $\varepsilon^{-1}x$ by $o(\varepsilon^{-1})$ is approximately described by a SDMF $\hat{F}(t; x, \cdot)$ and establish the connections between the evolved profile $\{\hat{F}(t; x, \cdot)\}$ and the initial one $\{\hat{F}(x, \cdot)\}$.

Our main result is that the family $\{\hat{F}(t, x, \cdot)\}$ satisfies the linear equation

$$\frac{\partial}{\partial t} \hat{F}(t; x, \theta) = A(\theta) \frac{\partial}{\partial x} \hat{F}(t; x, \theta), \qquad \theta \in [-\pi, \pi), \qquad t, x \in \mathbb{R}^1, \qquad t \neq 0 \tag{1.4}$$

where

$$A(\theta) = i\omega'(\theta) \begin{pmatrix} 0 & -1/\omega(\theta) \\ \omega(\theta) & 0 \end{pmatrix} \tag{1.5}$$

(the group velocity matrix). For any x and nonzero t the matrix function $\hat{F}(t; x, \cdot)$ satisfies the equilibrium condition. The limiting value $\hat{F}(0; x, \cdot)$ is connected with the initial matrix $\hat{F}(x, \cdot)$ by

$$\hat{F}^{1,1}(0; x, \theta) = \tfrac{1}{2}\{\hat{F}^{1,1}(x, \theta) + [1/\omega(\theta)^2]\, \hat{F}^{2,2}(x, \theta)\} \tag{1.6a}$$

$$\hat{F}^{1,2}(0; x, \theta) = i \operatorname{Im} \hat{F}^{1,2}(x, \theta) \tag{1. 6b}$$

$$\hat{F}^{2,1}(0; x, \theta) = i \operatorname{Im} \hat{F}^{2,1}(x, \theta) \tag{1.6c}$$

$$\hat{F}^{2,2}(0; x, \theta) = \tfrac{1}{2}\{\omega(\theta)^2\, \hat{F}^{1,1}(x, \theta) + \hat{F}^{2,2}(x, \theta)\} \tag{1.6d}$$

Notice that the map $\hat{F}(x, \cdot) \mapsto \hat{F}(0; x, \cdot)$ realizes the "projection" of the initial SDMF $\hat{F}(x, \cdot)$ onto the subspace of SDMFs satisfying the equilibrium condition.

The solution of (1.4), (1.6a–d) is given by the formulas

$$\begin{aligned} \hat{F}^{1,1}(t; x, \theta) = {} & \tfrac{1}{4}\{\hat{F}^{1,1}[x + \omega'(\theta)t, \theta] + \hat{F}^{1,1}[x - \omega'(\theta)t, \theta]\} \\ & + [1/4\omega(\theta)^2]\{\hat{F}^{2,2}[x + \omega'(\theta)t, \theta] + \hat{F}^{2,2}[x - \omega'(\theta)t, \theta]\} \\ & + [1/2\omega(\theta)]\{\operatorname{Im} \hat{F}^{1,2}[x + \omega'(\theta)t, \theta] \\ & - \operatorname{Im} \hat{F}^{1,2}[x - \omega'(\theta)t, \theta]\} \end{aligned} \tag{1.7a}$$

$$\begin{aligned} \hat{F}^{1,2}(t; x, \theta) = {} & [i\omega(\theta)/4]\{\hat{F}^{1,1}[x + \omega'(\theta)t, \theta] - \hat{F}^{1,1}[x - \omega'(\theta)t, \theta]\} \\ & + (i/4\omega(\theta)]\{\hat{F}^{2,2}[x + \omega'(\theta)t, \theta] - \hat{F}^{2,2}[x - \omega'(\theta)t, \theta]\} \\ & + i/2\{\operatorname{Im} \hat{F}^{1,2}[x + \omega'(\theta)t, \theta] \\ & + \operatorname{Im} \hat{F}^{1,2}[x - \omega'(\theta)t, \theta]\} \end{aligned} \tag{1.7b}$$

$$\hat{F}^{2,1}(t; x, \theta) = -\hat{F}^{1,2}(t; x, \theta) \tag{1.7c}$$

$$\hat{F}^{2,2}(t; x, \theta) = \omega(\theta)^2 \hat{F}^{1,1}(t; x, \theta) \tag{1.7d}$$

Equation (1.4) (or equivalently formulas 1.7a–d) means that for any fixed $\theta \in [-\pi, \pi)$, $\hat{F}(x, \theta)$, regarded as function of $x \in \mathbb{R}^1$ (the space profile for the "harmonic" θ), is changed in time independently of other harmonics $\theta' \neq \theta$. Physically speaking, we obtain the macroscopic picture of independently moving "normal modes" indexed by the pairs (x, θ). A normal mode (x, θ) at the macroscopic time moment t is "moving" at points $[x \pm \omega'(\theta) t, \theta]$, transferring here the (transformed) matrix $\hat{F}(x, \theta)$ with the weight $\frac{1}{2}$.

It is clear from above that the main object of our analysis is linear transformations of the SDMF family $\{F(x, \cdot)\}$. For this reason the methods used here are reduced to a careful analysis of oscillating integrals. The one-dimensional case we consider is the simplest from the technical point of view; in many dimensions we expect that the picture is similar.

The paper is organized as follows. In Section 2 we discuss some preliminary facts concerning the model. Section 3 contains the proof of the main result. In Section 4 we discuss the connection of the hydrodynamics with the first integrals of the harmonic motion. Finally, Section 5 deals with examples of families $\{P^\varepsilon, \varepsilon > 0\}$ for which the above-mentioned conditions are fulfilled.

2. NOTATIONS AND PRELIMINARY RESULTS

We start this section with basic notations and notions (some of them already used in the preceding section). Throughout this paper we denote by $\mathbb{Z}^1$, $\mathbb{Z}^1_+$, $\mathbb{R}^1$, $\mathbb{R}^1_+$, and $\mathbb{C}^1$ the integer, nonnegative integer, real, nonnegative real, and complex numbers, respectively. The symbol $[x]$ denotes the integer part of $x \in \mathbb{R}^1$. The Fourier transform of an l_2-sequence $\{g_k, k \in \mathbb{Z}^1\}$ is denoted by $\hat{g}$

$$\hat{g}(\theta) = \sum_{k \in \mathbb{Z}^1} g_k \exp(ik\theta), \qquad \theta \in [-\pi, \pi) \tag{2.1}$$

The notation $\Lambda \dot{\subset} \mathbb{Z}^1$ means that Λ is a bounded subset of $\mathbb{Z}^1$ and, as usual, Λ^c is the complementary set of Λ.

Hereafter we shall frequently deal with expressions written as sums, each term of which requires a separate analysis. If this is the case, we denote a single term by the corresponding formula number indexed by its order number. For example, $(3.28)_2$ denotes the second term in (3.28). If one is speaking on an equality, we usually have in mind its right-hand side. The possible deviations from this rule are specified in the text.

We consider a double-infinite system of classical one-dimensional oscillators. It is assumed that the jth oscillator has the equilibrium position at the point j, $j \in \mathbb{Z}^1$. We denote by $q_j \in \mathbb{R}^1$ and $p_j \in \mathbb{R}^1$ the displacement from the equilibrium position and momentum for the jth oscillator, respectively. The vector $(q_j, p_j) \in \mathbb{R}^2$ is denoted by y_j, $j \in \mathbb{Z}^1$. In many formulas it will be convenient to use an alternative notation: $q_j = y_j^1$, $p_j = y_j^2$.

Given $\Lambda \subseteq \mathbb{Z}^1$, we denote by $\mathscr{X}(\Lambda)$ the Cartesian product $(\mathbb{R}^2)^\Lambda$. This will be the phase space for the system of oscillators labeled by indices $j \in \Lambda$. A point $\mathbf{y}^{(\Lambda)} \in \mathscr{X}(\Lambda)$ is a sequence $\{y_j, j \in \Lambda\}$ of vectors $y_j \in \mathbb{R}^2$. If Λ is bounded, $\mathscr{X}(\Lambda)$ may be regarded as the finite-dimensional Hilbert space with the usual scalar product.

For a general Λ we endow $\mathscr{X}(\Lambda)$ with the product topology. $\mathscr{X}(\Lambda)$ becomes a Polish space. Denote by $\mathscr{B}(\Lambda)$ the Borel σ algebra of subsets of $\mathscr{X}(\Lambda)$: $\mathscr{B}(\Lambda)$ is generated by functions

$$\mathbf{y}^{(\Lambda)} \in \mathscr{X}(\Lambda) \mapsto y_j^\gamma, \ j \in \Lambda, \qquad \gamma = 1, 2 \tag{2.2}$$

If there will be no confusion, we denote the function (2.2) by y_j^γ (or q_j and p_j) as well; correspondingly, the vector function (y_j^1, y_j^2) will be denoted by y_j and the norm $[(y_j^1)^2 + (y_j^2)^2]^{1/2}$ by $\|y_j\|$. If Λ is bounded, we denote by $\mathscr{L}_\Lambda$ the Lebesgue measure on $[\mathscr{X}(\Lambda), \mathscr{B}(\Lambda)]$.

Given $\Lambda_1, \Lambda_2 \subseteq \mathbb{Z}^1$ where $\Lambda_1 \subset \Lambda_2$ and $\mathbf{y}^{(\Lambda_2)} \in \mathscr{X}(\Lambda_2)$, we denote by $(\mathbf{y}^{(\Lambda_2)})_{\Lambda_1}$ the restriction of $\mathbf{y}^{(\Lambda_2)}$ onto Λ_1. Given $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^1$ and $\mathbf{y}^{(\Lambda_i)} \in \mathscr{X}(\Lambda_i)$, $i = 1, 2$, where $\Lambda_1 \cap \Lambda_2 = \varnothing$, we denote by $\mathbf{y}^{(\Lambda_1)} \vee \mathbf{y}^{(\Lambda_2)}$ the sequence $\{y_j, j \in \Lambda_1 \cup \Lambda_2\}$ determined by

$$(\mathbf{y}^{(\Lambda_1)} \vee \mathbf{y}^{(\Lambda_2)})_{\Lambda_i} = \mathbf{y}^{(\Lambda_i)}, \qquad i = 1, 2$$

Sometimes we consider displacement sequences $\{q_j, j \in \Lambda\}$ or $\{y_j^1, j \in \Lambda\}$, denoting them by $\mathbf{q}^{(\Lambda)}$ or $(\mathbf{y}^1)^{(\Lambda)}$. The configuration space for the oscillator system in Λ is the Cartesian product $\mathbb{R}^\Lambda$; we denote it by $\mathscr{X}^1(\Lambda)$.

In the case $\Lambda = \mathbb{Z}^1$ the index (Λ) is omitted from the notations we introduced above; the same convention is used in all that follows. The set $\mathscr{X}$ is the phase space for the whole oscillator system under consideration. Given $\Lambda \subset \mathbb{Z}^1$, we can introduce the σ subalgebra of $\mathscr{B}$ generated by (vector) functions y_j, $j \in \Lambda$. There exists a natural isomorphism between this σ algebra and $\mathscr{B}(\Lambda)$; we shall not distinguish these two σ algebras using the same notation $\mathscr{B}(\Lambda)$ and identifying the corresponding sets.

A function $f: \mathscr{X} \to \mathbb{R}^1$ is called cylindrical if it is $\mathscr{B}(\Lambda)$-measurable for some $\Lambda \dot{\subset} \mathbb{Z}^1$.

In the usual way one defines the action $S_j: \mathscr{X} \to \mathscr{X}$, $j \in \mathbb{Z}^1$ of the group of space translations. By the same symbol S_j we denote the induced action on sets, functions, and measures.

Definition 2.1. A state P of the infinite oscillator system is a probability measure on the measurable space $(\mathscr{X}, \mathscr{B})$. The space of states is endowed with the topology of the vague convergence. A state P is called translationally invariant if

$$S_j P = P$$

The Rosenblatt mixing coefficient α_P of a state P is defined by

$$\alpha_P(h) = \sup_{k \in \mathbb{Z}^1} \sup_{\substack{A \in \mathscr{B}((-\infty, k]) \\ B \in \mathscr{B}([k+h, \infty))}} |P(A \cap B) - P(A)\, P(B)|, \quad h \in \mathbb{Z}^1 \tag{2.3}$$

The expectation w.r.t. P is denoted by $\langle \ \rangle_P$. Throughout this paper we consider states P with mean value 0, i.e.; $\langle y_j \rangle_P = 0$, $j \in \mathbb{Z}^1$. The covariance $\langle y_j^\gamma y_{j'}^{\gamma'} \rangle_P$, $j, j' \in \mathbb{Z}^1$, will play a special role in our analysis.

If a state P is translationally invariant (in the wide sense), the covariance (whenever it exists) depends on $j' - j$

$$\langle y_j^\gamma y_{j'}^{\gamma'} \rangle_P = (F_P^{\gamma,\gamma'})_{j'-j}, \qquad j, j' \in \mathbb{Z}^1, \qquad \gamma, \gamma' = 1, 2$$

Thereby we obtain the sequence of covariance matrices $F_P = \{(F_P)_k, k \in \mathbb{Z}^1\}$

$$(F_P)_k = \begin{pmatrix} (F_P^{1,1})_k & (F_P^{1,2})_k \\ (F_P^{2,1})_k & (F_P^{2,2})_k \end{pmatrix}$$

If $(F_P^{\gamma,\gamma'})_k$, $k \in \mathbb{Z}^1$, γ, $\gamma' = 1, 2$ are l_2-sequences, then, by passing to the Fourier transform, one introduces the spectral density matrix function

$$\hat{F}_P(\theta) = \begin{pmatrix} \hat{F}_P^{1,1}(\theta) & \hat{F}_P^{1,2}(\theta) \\ \hat{F}_P^{2,1}(\theta) & \hat{F}_P^{2,2}(\theta) \end{pmatrix}$$

As already mentioned, the diagonals $\hat{F}_P^{1,1}$, $\hat{F}_P^{2,2}$ are nonegative even functions and off-diagonals $\hat{F}_P^{1,2}$, $\hat{F}_P^{2,1}$ obey

$$\hat{F}_P^{1,2}(-\theta) = \overline{\hat{F}_P^{1,2}(\theta)} = \overline{\hat{F}_P^{2,1}(-\theta)} = \hat{F}_P^{2,1}(\theta), \qquad \theta \in [-\pi, \pi)$$

Notice in addition that for any θ the matrix $\hat{F}_P$ is positively semidefinite.

Definition 2.2. A state P is called Gaussian if for any $\Lambda \dot{\subset} \mathbb{Z}^1$ the joint probability distribution of vectors y_j, $j \in \Lambda$, is Gaussian. In our case it will be uniquely determined by its covariance.

From now on we shall always consider Gaussian states which are translationally invariant and have l_2-covariance sequences $\{(F_P^{\gamma,\gamma'})_k, k \in \mathbb{Z}^1\}$. Every such state P has the SDMF $\hat{F}_P$, and the map $P \mapsto \hat{F}_P$ is the one-to-one correspondence between the states and matrix functions $\hat{F}_P$ satisfying the above-listed conditions (see, e.g., Ref. 9a).

In particular, there are known necessary and sufficient conditions formulated in terms of the SDMF $\hat{F}_P$ of a Gaussian state P for $\alpha_P(h)$ to vanish as $h \to \infty$, and moreover for power or exponential decrease of $\alpha_P(h)$, see [9b]. Every Gaussian state P with l_2-covariance sequences $\{(F_P^{\gamma,\gamma'})_k, k \in \mathbb{Z}^1\}$ is ergodic.[9a]

Let a sequence $\Phi = \{\Phi^{(\Lambda)}, \Lambda \dot{\subset} \mathbb{Z}^1\}$ be given where $\Phi^{(\Lambda)}$ is a real measurable function

$$\mathbf{y}^{(\Lambda)} \in \mathscr{X}(\Lambda) \mapsto \Phi^{(\Lambda)}(\mathbf{y}^{(\Lambda)})$$

which is assumed to be even in the sense that $\Phi^{(\Lambda)}(\mathbf{y}^{(\Lambda)}) = \Phi^{(\Lambda)}(-\mathbf{y}^{(\Lambda)})$. Given $\Lambda \dot{\subset} \mathbb{Z}^1$ and $\mathbf{y}^{(\Lambda)} \in \mathscr{X}(\Lambda)$, we denote

$$\Psi_\Phi^{(\Lambda,\Lambda^c)}(\mathbf{y}^{(\Lambda)} \mid \mathbf{y}^{(\Lambda^c)}) = \lim_{N \to \infty} \sum_{\substack{\bar{\Lambda} \subseteq [-N,N] \\ \bar{\Lambda} \cap \Lambda \neq \varnothing}} \Phi^{(\bar{\Lambda})}[(\mathbf{y}^{(\Lambda)})_{\Lambda \cap \bar{\Lambda}} \vee (\mathbf{y}^{(\Lambda^c)})_{\Lambda^c \cap \bar{\Lambda}}] \tag{2.4}$$

If one interpretes Φ as a "potential of interaction," then the value $\Psi_\Phi^{(\Lambda,\Lambda^c)}(\mathbf{y}^{(\Lambda)} \mid \mathbf{y}^{(\Lambda^c)})$ gives the full "energy" of the system of oscillators in Λ whose displacements and momenta are given by vectors y_j, $j \in \Lambda$, in the external "field" generated by the oscillators outside Λ with displacement momentum vectors y_k, $k \in \Lambda$. In a similar way, by changing Λ^c with $\bar{\Lambda}$ in (2.4) we can define the energy $\Psi^{(\Lambda,\bar{\Lambda})}$ for any pair of disjoint sets $\Lambda \dot{\subset} \mathbb{Z}^1$, $\bar{\Lambda} \subset \mathbb{Z}^1$ (if $\bar{\Lambda}$ is bounded, then the limit on the right-hand side of (2.4) is not necessary). We define also the "proper" energy of the oscillator system in Λ by setting

$$\Psi_\Phi^{(\Lambda)}(\mathbf{y}^{(\Lambda)}) = \sum_{\bar{\Lambda} \subseteq \Lambda} \Phi^{(\bar{\Lambda})}[(\mathbf{y}^{(\Lambda)})_{\bar{\Lambda}}] \tag{2.5a}$$

We notice that (2.4) admits the following inversion

$$\Phi^{(\Lambda)}(\mathbf{y}^{(\Lambda)}) = \sum_{\bar{\Lambda} \subseteq \Lambda} (-1)^{\operatorname{Card}(\Lambda \setminus \bar{\Lambda})} \Psi_\Phi^{(\bar{\Lambda})}[(\mathbf{y}^{(\Lambda)})_{\bar{\Lambda}}] \tag{2.5b}$$

As usually, we say that the potential $\Phi = \{\Phi^{(\Lambda)}, \Lambda \dot{\subset} \mathbb{Z}^1\}$ is translationally invariant if for any $\Lambda \dot{\subset} \mathbb{Z}^1$ and $j \in \mathbb{Z}^1$

$$\Phi^{(\Lambda)}(\mathbf{y}^{(\Lambda)}) = \Phi^{(\Lambda+j)}(S_j \mathbf{y}^{(\Lambda)}), \qquad \mathbf{y}^{(\Lambda)} \in \mathscr{X}(\Lambda)$$

where $\Lambda + j = \{k \in \mathbb{Z}^1 : k - j \in \Lambda\}$ (this notation will be used below) and for $\mathbf{y}^{(\Lambda)} = \{y_l, l \in \Lambda\}$

$$S_j \mathbf{y}^{(\Lambda)} = \{y_l'; y_l' = y_{l-j}, l \in \Lambda + j\}$$

Definition 2.3. P is called a Gibbs state with the potential Φ if for any $\Lambda \dot{\subset} \mathbb{Z}^1$ (1) the limit $\Psi_\Phi[\mathbf{y}^{(\Lambda)} \mid (\mathbf{y})_{\Lambda^c}]$ exists for $(\mathscr{L}_\Lambda \times P)$-a.a. pairs $(\mathbf{y}^{(\Lambda)}, \mathbf{y}) \in \mathscr{X}(\Lambda) \times \mathscr{X}$ and the integral

$$\Xi_\Lambda^\Phi[(\mathbf{y})_{\Lambda^c}] = \int_{\mathscr{X}(\Lambda)} \mathscr{L}_\Lambda(d\mathbf{y}'^{(\Lambda)}) \exp\{-\Psi_\Phi^{(\Lambda,\Lambda^c)}[\mathbf{y}'^{(\Lambda)} \mid (\mathbf{y})_{\Lambda^c}]\} \tag{2.6a}$$

is finite for P-a.a. $\mathbf{y}$. The restriction on $\mathscr{B}(\Lambda)$ of the conditional probability measure $P[\cdot \mid \mathscr{B}(\Lambda^c)](\mathbf{y})$ for P-a.a. $\mathbf{y} \in \mathscr{X}$ is given by

$$P[A \mid \mathscr{B}(\Lambda^c)](\mathbf{y}) = \Xi^\Phi[(\mathbf{y})_{\Lambda^c}]^{-1} \int_A \mathscr{L}_\Lambda(d\mathbf{y}'^{(\Lambda)}) \exp\{-\Psi_\Phi^{(\Lambda,\Lambda^c)}[\mathbf{y}'^{(\Lambda)} \mid (\mathbf{y})_{\Lambda^c}]] \tag{2.6b}$$

where $\mathscr{L}_\Lambda$ is the Lebesgue measure in $\mathscr{X}(\Lambda)$.

After imposing certain restriction on Φ one can prove the existence and uniqueness (in some sense) of a Gibbs state with the potential Φ. See Section 5 for further details. A Gaussian state P with the covariance matrix sequence $\{(F_P)_k,\ k \in \mathbb{Z}^1\}$ is a Gibbs state with the quadratic potential $\Phi = \Phi_P$ of the form

$$\Phi^{(\Lambda)}(\mathbf{y}^{(\Lambda)}) = \begin{cases} \dfrac{1}{2} \sum\limits_{\gamma,\gamma'=1,2} (F_P^{-1})_0^{\gamma,\gamma'} y_j^\gamma y_j^{\gamma'}, & \text{if } \operatorname{Card} \Lambda = 1 \\ \qquad \Lambda = \{j\}, \quad \mathbf{y}^{(\Lambda)} = \{y_j\} & \\ \sum\limits_{\gamma,\gamma'=1,2} (F_P^{-1})_{j'-j}^{\gamma,\gamma'} y_j^\gamma y_{j'}^{\gamma'}, & \text{if } \operatorname{Card} \Lambda = 2, \\ \qquad \Lambda = \{j, j'\}, \quad \mathbf{y}^{(\Lambda)} = \{y_j, y_{j'}\} & \\ 0, \quad \text{if } \operatorname{Card} \Lambda \geqslant 3 & \end{cases} \tag{2.7}$$

Here the sequences $\{(F_P^{-1})_k^{\gamma,\gamma'},\ k \in \mathbb{Z}^1\}$, $\gamma, \gamma' = 1, 2$ are given by

$$(F_P^{-1})_k^{\gamma,\gamma'} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta [\hat{F}_P^{\gamma,\gamma'}(\theta)]^{-1} \exp(ik\theta), \qquad k \in \mathbb{Z}^1 \tag{2.8}$$

(if the integral makes sense).

Sometimes it is convenient to relate the Gibbs state P to the energy Ψ_Φ rather than to the potential Φ; this will be called Gibbs state with the energy function Ψ.

The harmonic dynamics of the oscillator system in $\Lambda \subseteq \mathbf{Z}^1$ is given by the solution of the Cauchy problem

$$\begin{aligned} \dot{q}_j(t) &= p_j(t) \\ \dot{p}_j(t) &= -\sum_{j' \in \Lambda} V(j'-j)\, q_{j'}(t), \qquad t \in \mathbb{R}^1, \quad j \in \Lambda \end{aligned} \tag{2.9}$$

$$\{[q_j(0), p_j(0)], j \in \Lambda\} = \mathbf{y}(\Lambda) \tag{2.10}$$

Here $V\colon \mathbb{Z}^1 \to \mathbb{R}^1$ is a fixed real function defining the harmonic interaction of oscillators. If Λ is bounded then the system (2.9) is finite and can be written in the hamiltonian form with the Hamiltonian

$$H^{(\Lambda)} = \frac{1}{2}\left[\sum_{j \in \Lambda} p_j^2 + \sum_{j,j' \in \Lambda} V(j'-j)\, q_j q_{j'}\right] \tag{2.11}$$

In the case where Λ is unbounded, (2.9) becomes infinite and $H^{(\Lambda)}$ is, in general, a formal expression. However, the right-hand side of (2.9) will converge whenever $V(k)$ decreases quickly enough and $q_k(t)$ does not increase too much as $k \to \pm\infty$. We shall impose the following conditions on V taken from [8].

(i) V is an even real function: $V(k) = V(-k)$, $k \in \mathbb{Z}^1$.

(ii) V has a compact support: $V(k) = 0$ for $k \geqslant k_0$ where $k_0 > 0$ is a constant.

Conditions (i), (ii) guarantee that $\hat{V}$ is a real analytic even function on $[-\pi, \pi)$.

(iii) V is positively definite: $\min_{\theta \in [-\pi,\pi)} \hat{V}(\theta) > 0$.

Condition (iii) allows us to introduce the positive even analytic function

$$\omega(\theta) = \hat{V}(\theta)^{1/2}, \qquad \theta \in [-\pi, \pi) \tag{2.12}$$

which plays an important role in our analysis. In what follows it will be convenient to use the function ω rather than V. For instance, the next condition reads as

(iv) ω is nondegenerate: the set

$$\{\theta\colon \omega''(\theta) = \omega'''(\theta) = 0\}$$

is empty.

Notice that conditions (ii) and (iii) imply that the set

$$\{\theta : \omega''(\theta) = 0\}$$

is finite.

Conditions (ii), (iv) play a technical role and way be weakened. For instance, exponential decay of $V(k)$ may be easily replaced by an inverse power. However, we should retain the hypothesis that the set of critical points of ω should be finite for the subsequent analysis of oscillating integrals. Instead of (iv) one may require that the set

$$\{\theta : \omega''(\theta) = \cdots \omega^{(m)}(\theta) = 0\} = \varnothing$$

for some value of $m \in \mathbb{Z}^1_+$; see Ref. 10. However, technical difficulties increase tremendously.

The system of differential equations (2.9) is linear

$$(d/dt)\, \mathbf{y}^{(\Lambda)}(t) = B^{(\Lambda)} \mathbf{y}^{(\Lambda)}(t) \tag{2.13}$$

whence we obtain the formula for the solution of (2.9), (2.10)

$$\mathbf{y}^{(\Lambda)}(t) = \exp(tB^{(\Lambda)})\, \mathbf{y}^{(\Lambda)}(0) \tag{2.14}$$

Here $B^{(\Lambda)}$ is the linear operator $\mathscr{X}(\Lambda) \to \mathscr{X}(\Lambda)$ given by

$$(B^{(\Lambda)} \mathbf{y}^{(\Lambda)})_j = \left[y_j^2,\ - \sum_{j' \in \Lambda} V(j' - j)\, y_{j'}^1 \right], \qquad j \in \Lambda \tag{2.15}$$

If Λ is a bounded set, then the operator $B^{(\Lambda)}$ is bounded in $\mathscr{X}(\Lambda)$ and the formula (2.13) gives the unique solution to (2.9), (2.10). In the case Λ is unbounded one needs a more detailed analysis. We restrict our attention to $\Lambda = \mathbb{Z}^1$. Given $m \in \mathbb{R}^1_+$, denote

$$\mathscr{X}'_m = \{\mathbf{y} \in \mathscr{X} : \sup_{j \in \mathbb{Z}^1} \| y_j \|^2 / (j^2 + 1)^m < \infty\} \tag{2.16}$$

Theorem 2.1 (see [11]). For any $\mathbf{y} \in \mathscr{X}'_m$, $m \in \mathbb{R}^1_+$, there exists the unique solution $\mathbf{y}(t)$, $t \in \mathbb{R}^1$ to the problem (2.9), (2.10) such that $\mathbf{y}(t) \in \mathscr{X}'_m$ for all t. This solution is given by the linear formula

$$y_j^\gamma(t) = \sum_{j' \in \mathbb{Z}^1} \sum_{\gamma' = 1,2} u_{j'-j}^{\gamma,\gamma'}(t)\, y_{j'}^{\gamma'}, \qquad \gamma = 1, 2 \tag{2.17}$$

The coefficients $u_k^{\gamma,\gamma'}(t)$ are defined by

$$u_k^{1,1} = u_k^{2,2}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \cos[\omega(\theta) t] \exp(-ik\theta) \tag{2.18}$$

$$u_k^{1,2}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta [1/\omega(\theta)] \sin[\omega(\theta) t] \exp(-ik\theta) \tag{2.18b}$$

$$u_k^{2,1} = -\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, \omega(\theta) \sin[\omega(\theta) t] \exp(-ik\theta) \tag{2.18c}$$

the function ω being defined in (2.12).

Remark. Let $\mathscr{Y}$ denote the l_2-subspace of $\mathscr{X}$. Conditions (i)–(iii) imply that the generator B (see 2.13, 2.15) is bounded on $\mathscr{Y}$. The operators $U(t)$ with the matrix elements $U(t)_{j,j'}^{\gamma,\gamma'} = u_{j'-j}^{\gamma,\gamma'}(t)$ are nothing but $\exp(tB)$. Passing to the Fourier transform we obtain that the operator $\hat{B}$ in $\mathscr{L}_2([-\pi, \pi)) \oplus \ell_2([-\pi, \pi))$ is the operator of multiplication by the matrix function

$$\hat{B}(\theta) = \begin{pmatrix} 0 & 1 \\ -\omega(\theta)^2 & 0 \end{pmatrix} \tag{2.19}$$

Correspondingly, $\hat{U}(t)$, $t \in \mathbb{R}^1$ is the operator of multiplication by the matrix function

$$\hat{U}(t, \theta) = \begin{pmatrix} \cos[\omega(\theta) t] & [1/\omega(\theta)] \sin[\omega(\theta) t] \\ -\omega(\theta) \sin[\omega(\theta) t] & \cos[\omega(\theta) t] \end{pmatrix} \tag{2.20}$$

Denote

$$\mathscr{X}' = \bigcup_{m>0} \mathscr{X}'_m \tag{2.21}$$

It is not hard to check that $\mathscr{X}' \in \mathscr{B}$. The time translations

$$T_t : \mathbf{y} \in \mathscr{X}' \mapsto \mathbf{y}(t), \qquad t \in \mathbb{R}^1 \tag{2.22}$$

form a continuous group of measurable transformations of $\mathscr{X}'$ onto itself. The induced action of this group on sets and functions is denoted by the same symbol T_t.

Definition 2.4. Let a state P be concentrated on the set $\mathscr{X}'$

$$P(\mathscr{X}') = 1 \tag{2.23}$$

The harmonic time evolution of the state P is the family of states $\{P_t, t \in \mathbb{R}^1\}$ given by

$$P_t(A) = P[T_{-t}(A \cap \mathscr{X}')], \qquad A \in \mathscr{B} \tag{2.24}$$

In the next part of this section we shall formulate assertions concerning some properties of the harmonic time evolution defined in (2.24). These assertions (Propositions 2.2–2.4 and Theorem 2.5) are taken from Ref. 8. For convenience we sometimes use simplified formulations.

A convenient sufficient condition for (2.23) is given in the following assertion.

Proposition 2.2. Let P be a state such that for all $j \in \mathbb{Z}^1$

$$\langle \| y_j \|^2 \rangle_P \leqslant K(1 + j^2)^m \tag{2.25}$$

where $K>0$ and $m>0$ are constants. Then P is concentrated on $\mathscr{X}'$ and for any $t\in\mathbb{R}^1$ and all $j\in\mathbb{Z}^1$

$$\langle \| y_j\|^2\rangle_{P_t}\leqslant K(t)(1+j^2)^m \tag{2.26}$$

with $K(t)>0$ being a new constant depending on t.

The covariance $\langle y_h^\gamma y_k^{\gamma'}\rangle_{P_t}$ is given by

$$\langle y_h^\gamma y_k^{\gamma'}\rangle_{P_t}=\sum_{\delta,\delta'=1}^{2}\sum_{l,l'\in\mathbb{Z}^1} u_{h-l}^{\gamma,\delta}(t)\, u_{k-l'}^{\gamma',\delta'}(t)\langle y_l^\delta y_{l'}^{\delta'}\rangle_P \tag{2.27}$$

An important question is to investigate the time invariant states, i.e., states P for which

$$P_t=P,\qquad t\in\mathbb{R}^1 \tag{2.28}$$

Proposition 2.3. Let P be a Gaussian state with the SDMF $\hat{F}_P$. Then the following statements are equivalent

(i) P is time invariant

(ii) the matrix $\hat{F}_P$ is of the form

$$\hat{F}_P(\theta)=\begin{pmatrix}\hat{g}(\theta) & \hat{h}(\theta)\\ -\hat{h}(\theta) & \omega(\theta)^2\,\hat{g}(\theta)\end{pmatrix} \tag{2.29}$$

where $\hat{g}$ is an even nonnegative function and $\hat{h}$ an odd purely imaginary function on $[-\pi,\pi)$.

It will be convenient to characterize time-invariant Gaussian states from "Gibbsian" point of view. For simplicity, let us assume now that V has compact support. Given $\Lambda\dot{\subset}\mathbb{Z}^1$ and $\mathbf{y}^{(\Lambda)}\in\mathscr{X}(\Lambda)$, denote

$$E(h,j;\mathbf{y}^{(\Lambda)})=\tfrac{1}{2}[p_h p_j+q_h(\mathbb{V}\mathbf{q}^{(\Lambda)})_j],\qquad h,j\in\Lambda \tag{2.30}$$

$$A(h,j;\mathbf{y}^{(\Lambda)})=\tfrac{1}{2}(q_h p_j-p_h q_j),\qquad h,j\in\Lambda \tag{2.31}$$

Here $\mathbb{V}=\mathbb{V}_\Lambda$ denotes the operator in $\mathscr{X}^1(\Lambda)$ which reads as

$$(\mathbb{V}\mathbf{q}^{(\Lambda)})_j=\sum_{j'\in\Lambda}V(j'-j)q_{j'},\qquad j\in\Lambda,\quad \mathbf{q}^{(\Lambda)}\in\mathscr{X}^1(\Lambda)$$

By the same formulas, replacing $\mathbf{y}^{(\Lambda)}\in\mathscr{X}(\Lambda)$ by $\mathbf{y}\in\mathscr{X}$ one may introduce the quantities $E(h,j;\mathbf{y})$ and $A(h,j;\mathbf{y})$, $h,j\in\mathbb{Z}^1$.

Proposition 2.4. Let P be a Gaussian state with l_1-covariance matrix sequence $\{(F_P)_k,\ k\in\mathbb{Z}^1\}$ and nondegenerate SDMF $\hat{F}_P$: $\min_{\theta\in[-\pi,\pi)}\det\hat{F}_P(\theta)>0$. Then P is time-invariant iff the energy function

$\Psi = \{\Psi^{(\Lambda)}, \Lambda \dot{\subset} \mathbb{Z}^1\}$ corresponding to the quadratic potential $\Phi = \{\Phi^{(\Lambda)}\}$ (see 2.7) is of the form

$$\Psi^{(\Lambda)}[\mathbf{y}^{(\Lambda)}] = \sum_{h,j \in \Lambda} [E(h, j; \mathbf{y}^{(\Lambda)}) \lambda_{j-h} + A(h, j; \mathbf{y}^{(\Lambda)}) \mu_{j-h}] \qquad (2.32)$$

where λ_k and μ_k, $k \in \mathbb{Z}^1$ are the Fourier coefficients of functions $\hat{f}_1$ and $\hat{f}_2$: $[-\pi, \pi) \to \mathbb{C}^1$, respectively, which may be represented as

$$\hat{f}_1 = 2 \frac{\hat{g}}{\omega^2 \hat{g}^2 + \hat{h}^2}, \qquad \hat{f}_2 = -2 \frac{\hat{h}}{\omega^2 \hat{g}^2 + \hat{h}^2} \qquad (2.33)$$

Here the functions $\hat{g}$ and $\hat{h}$ are as in Proposition 2.3 and the SDMF $\hat{F}_P$ of the state P is given by (2.29).

Formula (2.32) is interesting because of its connections with first integrals of the harmonic dynamics. Given $k \in \mathbb{Z}^1$ and $\mathbf{y} \in \mathscr{X}$, we set

$$e(k; \mathbf{y}) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N \leqslant h \leqslant N-k} E(h, h+k; \mathbf{y}_{[-N,N]}) \qquad (2.34)$$

$$a(k; \mathbf{y}) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N \leqslant h \leqslant N-k} A(h, h+k; \mathbf{y}_{[-N,N]}) \qquad (2.35)$$

whenever these limits exist. Let $\mathscr{X}^0$ be the set of $\mathbf{y} \in \mathscr{X}$ for which the limits (2.34), (2.35) exist for all $k \in \mathbb{Z}^1$. It is possible to check that $T_t \mathscr{X}^0 \subset \mathscr{X}^0$ and that for any $k \in \mathbb{Z}^1$, $t \in \mathbb{R}^1$ and $\mathbf{y} \in \mathscr{X}' \cap \mathscr{X}^0$

$$e(k; T_t \mathbf{y}) = e(k; \mathbf{y}), \qquad a(k; T_t \mathbf{y}) = a(k; \mathbf{y}) \qquad (2.36)$$

It is not hard to give conditions on a state P under which the equality $P(\mathscr{X}^0) = 1$ holds. For instance, it suffices to require existence of the limit of expectation values

$$\lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N \leqslant h \leqslant N-k} \langle E(h, h+k; \mathbf{y}_{[-N,N]}) \rangle_P \qquad (2.37)$$

$$\lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N \leqslant h \leqslant N-k} \langle A(h, h+k; \mathbf{y}_{[-N,N]}) \rangle_P \qquad (2.38)$$

together with existence and boundedness of expectations $\langle \| y_j \|^{4+\delta} \rangle_P$ for some $\delta > 0$ and appropriate decreasing of the mixing coefficient $\alpha_P(h)$ when $h \to \infty$ (see Theorem 2.5 below).

Notice that if P is time-invariant Gaussian with a nondegenerate spectral density matrix $\hat{F}_P$ of the form (2.29), then

$$\langle e(k, \cdot) \rangle_P = g_k, \qquad \langle a(k; \cdot) \rangle_P = h_k \qquad (2.39)$$

where g_k and h_k, $k \in \mathbb{Z}^1$, are the Fourier coefficients of functions $\hat{g}$ and $\hat{h}$, respectively.

We now formulate a result on the limit behavior of time-evolved states P_t, $t \in \mathbb{R}^1$, when $t \to \pm\infty$.

Theorem 2.5. Let P be a state such that

$$\sup_{j \in \mathbb{Z}^1} \langle \| y_j \|^{4+\delta} \rangle_P < \infty \tag{2.40}$$

and

$$\sum_{h \in \mathbb{Z}^1} h^2 \alpha_P(h)^{\delta/(4+\delta)} < \infty \tag{2.41}$$

for some $\delta > 0$. Then state P_t defined in (2.24) converges as $t \to \pm\infty$ to a Gaussian state G if for any $\gamma, \gamma' = 1, 2$ and $j, j' \in \mathbb{Z}^1$

$$\lim_{t \to \pm\infty} \langle y_j^\gamma y_{j'}^{\gamma'} \rangle = (F_G^{\gamma,\gamma'})_{j'-j} \tag{2.42}$$

Convergence in (2.42) may be verified in a variety of cases, including translationally invariant, periodic, and "almost" periodic initial states P.

In order to simplify the arguments below we shall use the statement of Theorem 2.5 in the particular case $\delta = 1$. The set of states which satisfy (2.40) and (2.41) with $\delta = 1$ is denoted by $\mathfrak{B}$.

Finally, concluding this section we shall formulate technical lemmas which will be repeatedly used below. For the proof of Lemma 2.6 see [8], Propositions A.2 and A.3, and Ref. 10, Lemma 2.2. The proof of Lemma 2.7 is based on arguments used in the proof of Theorem 3.1 from Ref. 8 (p. 132–133).

Lemma 2.6. Assume $f, g: [-\pi, \pi) \to \mathbb{R}^1$ are periodic analytic functions and the set

$$\{\theta: f''(\theta) = f'''(\theta) = 0\}$$

is empty. Let $u_y(t)$, $y, t \in \mathbb{R}^1$ be defined by

$$u_y(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, g(\theta) \exp[iy\theta + itf(\theta)]$$

Then the following assertions hold:

(i) There exists a constant $c' > 0$ (depending on $\min_{\theta: f''(\theta)=0} |f'''(\theta)|$) such that for all $t, y \in \mathbb{R}^1$

$$|u_y(t)| \leqslant c'(1 + |t|)^{-1/3} \tag{2.43}$$

(ii) For any $\alpha \in (\frac{1}{3}, 1]$, there exists a constant $c'' = c''(\alpha) > 0$ (depending on $\min_{\theta: f''(\theta)=0} |f'''(\theta)|$ as well) such that for all $t, y \in \mathbb{R}^1$ with $\min_{\theta: f''(\theta)=0} |y - tf'(\theta)| > |t|^\alpha$

$$|u_y(t)| \leqslant c''(1 + |t|)^{-(1/4)(1+\alpha)} \tag{2.44}$$

(iii) Now let $y \in \mathbb{Z}^1$ and $c > \max_{\theta \in [-\pi, \pi)} |f'(\theta)|$ be fixed. Then for any $\alpha > 0$ there exists a constant $c''' = c'''(\alpha) > 0$ such that for all $t \in \mathbb{R}'$, $y \in \mathbb{Z}^1$ with $|y| > c|t|$

$$|u_y(t)| \leqslant c'''(1 + |y|)^{-\alpha}$$

In particular, bounds (2.43), (2.44) are valid for the motion coefficient $u_k^{\gamma,\gamma'}(t)$, $t \in \mathbb{R}^1$, $k \in \mathbb{Z}^1$, $\gamma, \gamma' = 1, 2$, defined by (2.18a–c).

Lemma 2.7. For any fixed $c > 0$ the second moments are bounded uniformly in $P \in \mathfrak{B}$ with $\sup_{j \in \mathbb{Z}^1} \langle \| y_j \|^5 \rangle_P \leqslant c$, $t \in \mathbb{R}^1$ and $h, k \in \mathbb{Z}^1$.

Moreover, the functions $[E(h, h+k; \cdot)]^s$ and $[A(h, h+k; \cdot)]^{s'}$, $s, s' = 1, 2$, are "uniformly integrable": for every $\eta > 0$ there exists $b = b(\eta) > 0$ such that for all $P \in \mathfrak{B}$ with $\sup_{j \in \mathbb{Z}^1} \langle \| y_j \|^5 \rangle_P \leqslant c$, all $t \in \mathbb{R}^1$ and $h, k \in \mathbb{Z}^1$

$$\langle | [E(h, h+k; \cdot)]^s - [E^{(b)}(h, h+k; \cdot)]^s | \rangle_{P_t} < \eta$$

$$\langle | [A(h, h+k; \cdot)]^{s'} - [A^{(b)}(h, h+k; \cdot)]^{s'} | \rangle_{P_t} < \eta$$

Here

$$E^{(b)}(h, h+k; \cdot) \begin{cases} = E(h, h+k; \cdot), & \text{if } |E(h, h+k; \cdot)| < b \\ = 0, & \text{otherwise,} \end{cases}$$

and $A^{(b)}(h, h+k; \cdot)$ is defined in a similar way.

3. THE HYDRODYNAMIC LIMIT

In this section we treat the central issue of this paper. Consider a complex (2×2) matrix function $\hat{F}$ on $\mathbb{R}^1 \times [-\pi, \pi)$

$$\hat{F}(x, \theta) = \begin{pmatrix} \hat{F}^{1,1}(x, \theta) & \hat{F}^{1,2}(x, \theta) \\ \hat{F}^{2,1}(x, \theta) & \hat{F}^{2,2}(x, \theta) \end{pmatrix}, \qquad x \in \mathbb{R}^1, \quad \theta \in [-\pi, \pi) \tag{3.1}$$

with the following properties:

(A) For every fixed $x \in \mathbb{R}^1$ and $\gamma, \gamma' = 1,2$, the function $\hat{F}^{\gamma,\gamma'}(x, \theta)$ is bounded on $[-\pi, \pi)$ and the inverse Fourier transform

$$F_k^{\gamma,\gamma'}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-i\theta k)\, \hat{F}^{\gamma,\gamma'}(x, \theta)\, d\theta \tag{3.2}$$

satisfies the bounds

$$|F_k^{\gamma,\gamma'}(x)| \leqslant a_1 \exp(-a_0 |k|) \qquad k \in \mathbb{Z}^1$$

where a_0, a_1 are positive constants.

(B) For every fixed $x \in \mathbb{R}^1$ the diagonals $\hat{F}^{1,1}(x, \cdot)$, $\hat{F}^{2,2}(x, \cdot)$ are nonnegative even functions and the off-diagonals $\hat{F}^{1,2}(x, \cdot)$, $\hat{F}^{2,1}(x, \cdot)$ obey

$$\hat{F}^{1,2}(x, -\theta) = \overline{\hat{F}^{1,2}(x, \theta)} = \overline{\hat{F}^{2,1}(x, -\theta)} = \hat{F}^{2,1}(x, \theta) \qquad \theta \in [-\pi, \pi)$$

(C) For fixed $x \in \mathbb{R}^1$ and $\theta \in [-\pi, \pi)$ the matrix $\hat{F}(x, \theta)$ is positively semidefinite.

(D) For every $\theta \in [-\pi, \pi)$, $\hat{F}^{\gamma,\gamma'}(\cdot, \theta)$, $\gamma, \gamma' = 1, 2$ are C^1 functions and the functions

$$x \to \sup_{\theta \in [-\pi,\pi)} \max_{\gamma,\gamma'} \left(|\hat{F}^{\gamma,\gamma'}(x, \theta)|, \left| \frac{\partial}{\partial x} \hat{F}^{\gamma,\gamma'}(x, \theta) \right| \right)$$

are bounded uniformly on bounded intervals.

As noted above, conditions (B) guarantee that for any $x \in \mathbb{R}^1$, $\hat{F}(x, \cdot)$ is a SDMF for some (e.g., Gaussian) state. For this reason we call $\hat{F}$ a space SDMF profile. It will play the role of the initial data for the hydrodynamic equations we derive. The space SDMF profile describes the "macroscopic" structure of the family of initial states.

More precisely, we introduce the following definition.

Definition 3.1. Let a family of states be given in terms of a parameter $\varepsilon > 0$: $\{P^\varepsilon, \varepsilon > 0\}$. We call it a hydrodynamical family for $\hat{F}$ if:

(a) For any $\varepsilon > 0$ there exists an even integer N_ε such that:

(i) for all $w \in \mathbb{R}^1$ and $s, s' \in I_w$

$$|\langle y_s^\gamma y_{s'}^{\gamma'} \rangle_{P^\varepsilon} - F_{s-s'}^{\gamma,\gamma'}(\varepsilon w)| \leqslant a_1 \min[\exp(-a_0 |s - s'|), \varepsilon N_\varepsilon] \tag{3.3}$$

where a_0, a_1 are the constants figuring in (3.2) and

$$I_w = [w - \tfrac{1}{2} N_\varepsilon, w + \tfrac{1}{2} N_\varepsilon] \cap \mathbb{Z}^1 \tag{3.4}$$

(ii) $\lim_{\varepsilon \to 0} \varepsilon^{2/3} N_\varepsilon = +\infty$ and $\exists \alpha \in (\frac{1}{3}, \frac{2}{3})$ such that

$$\lim_{\varepsilon \to 0} N_\varepsilon \varepsilon^{(1/4)[\alpha + (7/3)]} = 0 \tag{3.5}$$

(b) For any $\varepsilon > 0$ and all $s, s' \in \mathbb{Z}^1$, $\gamma, \gamma' = 1, 2$

$$|\langle y_s^\gamma y_{s'}^{\gamma'} \rangle_{P^\varepsilon}| \leqslant a_1 \exp(-a_0 |s - s'|) \tag{3.6}$$

with the same constants a_0, a_1 as in (3.2).

Remark. Condition (3.3) means a kind of translation invariance on an intermediate scale; it could have been expressed in a more natural (and equivalent) way by

$$|\langle y_s^\gamma y_{s'}^{\gamma'} \rangle_{P^\varepsilon} - F_{s-s'}(\varepsilon s)| \leqslant a_1 \min[\exp(-a_0 |s - s'|), \varepsilon^\beta]$$

with a suitable $\beta \in (0, 1)$.

As in the following explicit calculations it will be technically useful to partition the microscopic lattice in intervals (blocks) we use the slightly more artificial formulation (3.3).

In Section 1 we gave a comment on conditions (a) and (b). In particular, from condition (a) it follows that for any $x \in \mathbb{R}^1$ and $l \in \mathbb{Z}^1$

$$\lim_{\varepsilon \to 0} \langle y_{[\varepsilon^{-1}w]}^\gamma y_{[\varepsilon^{-1}x]+l}^{\gamma'} \rangle_{P^\varepsilon} = F_l^{\gamma,\gamma'}(x) \qquad \gamma, \gamma' = 1, 2 \tag{3.7}$$

It is not hard to check as well that for any fixed $x \in \mathbb{R}^1$ the Fourier transform

$$\sum_{l \in \mathbb{Z}'} \langle y_{[\varepsilon^{-1}x]}^\gamma y_{[\varepsilon^{-1}x]+l}^{\gamma'} \rangle_{P^\varepsilon} \exp(il\theta) \tag{3.8}$$

converges to the function $\hat{F}^{\gamma,\gamma'}(x, \theta)$ uniformly on $[-\pi, \pi)$. The main result of this section is the following

Theorem 3.1. Let $\{P^\varepsilon, \varepsilon > 0\}$ be a hydrodynamical family of states for a space SDMF profile $\hat{F}$ which satisfies conditions (A–D) above. Then, for any $x \in \mathbb{R}^1$, $l \in \mathbb{Z}^1$, nonzero $t \in \mathbb{R}^1$ and $\gamma, \gamma' = 1, 2$ there exist the limits

$$F_l^{\gamma,\gamma'}(\varepsilon, x) = \lim_{\varepsilon \to 0} \langle y_{[\varepsilon^{-1}x]}^\gamma y_{[\varepsilon^{-1}x]+l}^{\gamma'} \rangle_{P^\varepsilon_{\varepsilon^{-1}t}} \tag{3.9}$$

Here $P_{\tilde{t}}^\varepsilon$, $\tilde{t} \in \mathbb{R}^1$ denotes the harmonic time evolution of the initial state P^ε (see 2.24). The Fourier transforms

$$\hat{F}^{\gamma,\gamma'}(t, x, \theta) = \sum_{l \in \mathbb{Z}'} F_l^{\gamma,\gamma'}(t, x) \exp(il\theta), \qquad \theta \in [-\pi, \pi) \tag{3.10}$$

are given by the formulas (1.7a–d).

Proof of Theorem 3.1. For the sake of brevity we assume that $t>0$ and consider relation (3.9) for $\gamma=\gamma'=1$; the other cases are treated in the same way. According to (2.27) we have

$$\langle y^1_{[\varepsilon^{-1}x]}\, y^1_{[\varepsilon^{-1}x]+l}\rangle_{P^\varepsilon_{\varepsilon^{-1}t}} = \sum_{\delta,\delta'=1,2}\ \sum_{n,n'\in\mathbb{Z}^1} u^{1,\delta}_{[\varepsilon^{-1}x]-n}(\varepsilon^{-1}t)\, u^{1,\delta'}_{[\varepsilon^{-1}x]+l-n'}(\varepsilon^{-1}t)\langle y^\delta_n y^{\delta'}_{n'}\rangle_{P^\varepsilon} \tag{3.11}$$

where the coefficients $u^{1,\delta}_k(t)$ are defined in (2.18a,b). It is convenient to introduce, for a fixed constant $c>\max_{\theta\in[-\pi,\pi)}|\omega'(\theta)|$, the following set

$$J_\varepsilon = J_\varepsilon(c,t) = \{\nu\in\mathbb{Z}^1 : I_{\nu N_\varepsilon}\subset[-\varepsilon^{-1}ct, \varepsilon^{-1}ct]\} \tag{3.12}$$

The first step in the proof is

Proposition 3.2. The following representation holds

$$\langle y^1_{[\varepsilon^{-1}x]}\, y^1_{[\varepsilon^{-1}x]+l}\rangle_{P^\varepsilon_{\varepsilon^{-1}t}} = \sum_{\delta,\delta'=1,2}\ \sum_{\nu,\nu'\in J_\varepsilon}\ \sum_{\substack{m\in I_{\nu N_\varepsilon}\\ m'\in I_{\nu' N_\varepsilon}}} [u^{1,\delta}_m(\varepsilon^{-1}t)\, u^{1,\delta'}_{m'+l}(\varepsilon^{-1}t) \cdot \langle y^\delta_{[\varepsilon^{-1}x]-m}\, y^{\delta'}_{[\varepsilon^{-1}x]-m'}\rangle_{P^\varepsilon}] + G_1(\varepsilon,t) \tag{3.13}$$

where $\lim_{\varepsilon\to 0}\varepsilon^{-b}G_1(\varepsilon,t)=0$ for any $b\geqslant 0$, $t\in\mathbb{R}'$.

The proof of Proposition 3.2 is an immediate consequence of Lemma 2.6 (ii). The next step in the proof of Theorem 3.1 is:

Proposition 3.3. The following representation holds

$$(3.13)_1 = \sum_{\delta,\delta'=1,2}\ \sum_{\nu\in J_\varepsilon}\ \sum_{m,m'\in I_{\nu N_\varepsilon}} u^{1,\delta}_m(\varepsilon^{-1}t)\, u^{1,\delta'}_{m'+l}(\varepsilon^{-1}t)\langle y^\delta_{[\varepsilon^{-1}x]-m}\, y^{\delta'}_{[\varepsilon^{-1}x]-m'}\rangle_{P^\varepsilon} + G_2(\varepsilon,t) \tag{3.14}$$

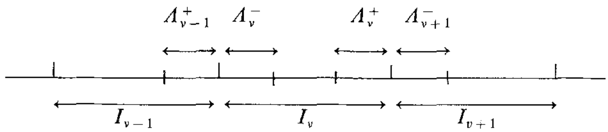
where $\lim_{\varepsilon\to 0} G_2(\varepsilon,t)=0$ for any $t\in\mathbb{R}'$.

Proof of Proposition 3.3. The "intertwining" term $G_2(\varepsilon,t)$ contains the external sums over pairs $\nu,\nu'\in J_\varepsilon$ with $\nu\neq\nu'$. The pairs ν,ν' with $|\nu'-\nu|\geqslant 2$ via Lemma 2.6(i) and relation (3.6), give the contribution of the order

$$a_1(\varepsilon N_\varepsilon)^{-1} N_\varepsilon \varepsilon^{2/3} \sum_{s\geqslant N_\varepsilon-|l|} \exp(-a_0 s)$$

which vanishes as $\varepsilon\to 0$ because of conditions (3.3) and (3.5).

In order to estimate the contribution of pairs of nearest neighbors v, v' we extract in every interval I_{vN_ε} the "boundary zones" $\Lambda^{\pm}_{vN_\varepsilon}$ of length $[N^{\lambda}_{\varepsilon}]$ where $\lambda \in (0, 1)$ is chosen (see figure below).



The addends

$$u^{1,\delta}_m(\varepsilon^{-1}t)\, u^{1,\delta'}_{m'+l}(\varepsilon^{-1}t) \langle y^{\delta}_{[\varepsilon^{-1}x]-m}\, y^{\delta'}_{[\varepsilon^{-1}x]-m'} \rangle_{P^\varepsilon}$$

where either $m \in I_{vN_\varepsilon} \setminus \Lambda^{\pm}_{vN_\varepsilon}$ or $m' \in I_{v'N_\varepsilon} \setminus \Lambda^{\mp}_{v'N_\varepsilon}$, $v' - v = \pm 1$ give the contribution of the order

$$a_1 (\varepsilon N_\varepsilon)^{-1} \varepsilon^{2/3} \sum_{s \geqslant [N^{\lambda}_{\varepsilon}]} \exp(-a_0 s)$$

while those with $m \in \Lambda^{\pm}_{vN_\varepsilon}$, $m' \in \Lambda^{\mp}_{v'N_\varepsilon}$, $v' - v = \pm 1$ give the contribution of the order

$$(\varepsilon N_\varepsilon)^{-1} \varepsilon^{2/3} N^{2\lambda}_{\varepsilon} = \varepsilon^{-1/3} N^{2\lambda - 1}_{\varepsilon}$$

By relations (3.3) and (3.5) we may choose λ in such a way that all bounds vanish when $\varepsilon \to 0$. Proposition 3.3 is proven. Now we use condition (3.3).

Proposition 3.4. The following representation holds

$$\begin{aligned}(3.14)_1 = \sum_{\delta,\delta'=1,2} \sum_{v \in J_\varepsilon} \sum_{m,m' \in I_{vN_\varepsilon}} & \{ u^{1,\delta}_m(\varepsilon^{-1}t)\, u^{1,\delta'}_{m'+l}(\varepsilon^{-1}t) \\ & \cdot F^{\delta,\delta'}_{m'-m}(\varepsilon[\varepsilon^{-1}x] - \varepsilon v N_\varepsilon)\} + G_3(\varepsilon, t) \qquad (3.15)\end{aligned}$$

and $\lim_{\varepsilon \to 0} G_3(\varepsilon, t) = 0$ for any $x, t \in \mathbb{R}^1$.

Proof of Proposition 3.4. According to (3.3), the difference between $F^{\delta,\delta'}_{m'-m}(\varepsilon[\varepsilon^{-1}x] - \varepsilon v N_\varepsilon)$ and $\langle y^{\delta}_{[\varepsilon^{-1}x]-m}\, y^{\delta'}_{[\varepsilon^{-1}x]-m'} \rangle_{P^\varepsilon}$ does not exceed in modulo

$$a_1 \min[\exp(-a_0 |m - m'|), \varepsilon N_\varepsilon]$$

At fixed m we sum this quantity over m'. Using Lemma 2.6(i) we obtain that the sum is less or equal to

$$\bar{a}_1 \varepsilon^{1/3} (\varepsilon N_\varepsilon + \varepsilon N_\varepsilon |\log \varepsilon N_\varepsilon|)$$

Finally, according to Lemma 2.6(ii) for all m for which

$$\min_{\theta:\omega''(\theta)=0} |m-\varepsilon^{-1}t\omega'(\theta)| > |t|^{\alpha}\varepsilon^{-\alpha}$$

the following bound holds

$$|u_m^{1,\delta}(\varepsilon^{-1}t)| < \mathbf{c}''\varepsilon^{(1/4)(1+\alpha)}$$

($\mathbf{c}''$ being independent on t as well).

Taking all this into account we estimate

$$|G_3(\varepsilon, t)| \leqslant \bar{\bar{a}}_1\varepsilon^{-1}N_\varepsilon \varepsilon N_\varepsilon(1+|\log \varepsilon N_\varepsilon|)\cdot(\varepsilon^{-\alpha+2/3}+N_\varepsilon \varepsilon^{1/3+(1/4)(1+\alpha)})$$

and see that the choice $\alpha \in (\frac{1}{3}, \frac{2}{3})$ guarantees that $\lim_{\varepsilon\to 0} G_3(\varepsilon, t)=0$.

We now move on to study the first sum in (3.15). For the sake of brevity consider a single term in the external sum $\sum_{\delta,\delta'}$ which corresponds to $\delta, \delta'=1$. Let us denote the corresponding sum $\sum_{v\in J_\varepsilon}\sum_{m,m'\in I_{vN_\varepsilon}}$ by $(3.15)_{1,1}$. Using formula (2.18a) and the fact that ω is an even function, we write

$$(3.15)_{1,1} = \frac{1}{4\pi^2}\sum_{v\in J_\varepsilon}\sum_{m,m'\in I_{vN_\varepsilon}}\int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\theta'\, e^{-i[m\theta-(m'-l)\theta']}$$
$$\cdot\cos[\omega(\theta)\,\varepsilon^{-1}t]\cos[\omega(\theta')\,\varepsilon^{-1}t]\, F_{m'-m}^{1,1}(\varepsilon[\varepsilon^{-1}x]-\varepsilon vN_\varepsilon) \tag{3.16}$$

Proposition 3.5. The following representation holds

$$(3.16) = \frac{1}{4\pi^2}\sum_{v\in J_\varepsilon}\sum_{m\in I_{vN_\varepsilon}}\int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\theta'\, e^{i[l\theta'+m(\theta'-\theta)]}$$
$$\cdot\cos[\omega(\theta)\varepsilon^{-1}t]\cos[\omega(\theta')\varepsilon^{-1}t]\,\hat{F}^{1,1}(\varepsilon[\varepsilon^{-1}x]-\varepsilon vN_\varepsilon, \theta') + G_4(\varepsilon_1 t) \tag{3.17}$$

where $\lim_{\varepsilon\to 0} G_4(\varepsilon, t)=0$ for any $t\in\mathbb{R}^1$.

Proof of Proposition 3.5. To prove the statement of Proposition 3.5 one should repeat in the reversed order the construction based on Propositions 3.2 and 3.3. For the sake of brevity we do not enter into details.

In the next step of the proof of Theorem 3.1 we consider the sum $(3.17)_1$. Performing the summation over m, we write

$$(3.17)_1 = \frac{1}{4\pi^2}\sum_{v\in J_\varepsilon}\int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\theta'\, e^{il\theta'}\frac{e^{i(v+1/2)(\theta'-\theta)N_\varepsilon}-e^{i(v-1/2)(\theta'-\theta)N_\varepsilon}}{e^{i(\theta'-\theta)}-1}$$
$$\cdot\cos[\omega(\theta)\varepsilon^{-1}t]\cos[\omega(\theta')\varepsilon^{-1}t]\,\hat{F}^{1,1}(\varepsilon[\varepsilon^{-1}x]-\varepsilon vN_\varepsilon, \theta') \tag{3.18}$$

By replacing variables $(\theta, \theta') \to (\theta - \varphi, \theta)$ and the 2π-periodicity of the inner integral, the right-hand side of (3.18) becomes

$$\frac{1}{4\pi^2} \sum_{\nu \in J_\varepsilon} \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\varphi \, e^{il\theta} \frac{e^{i(\nu + 1/2)N_\varepsilon \varphi} - e^{i(\nu - 1/2)N_\varepsilon \varphi}}{e^{i\varphi} - 1}$$
$$\cdot \cos[\omega(\theta)\varepsilon^{-1}t] \cos[\omega(\theta - \varphi)\varepsilon^{-1}t] \, \hat{F}^{1,1}(\varepsilon[\varepsilon^{-1}x] - \varepsilon\nu N_\varepsilon, \theta) \qquad (3.19)$$

Writing $\cos(\cdot) = \frac{1}{2}(e^{i(\cdot)} + e^{-i(\cdot)})$ we reduce the problem of evaluating the limit value of (3.19) to the problem of evaluating the limits of sums over $\nu \in J_\varepsilon$ of the four integrals

$$\frac{1}{16\pi^2} \int_{-\pi}^{\pi} d\theta \, e^{il\theta} e^{\pm i\omega(\theta)\varepsilon^{-1}t} \int_{-\pi}^{\pi} d\varphi \, \frac{e^{i(\nu + 1/2)N_\varepsilon \varphi} - e^{i(\nu - 1/2)N_\varepsilon \varphi}}{e^{i\varphi} - 1}$$
$$\cdot e^{\pm i\omega(\theta - \varphi)\varepsilon^{-1}t} \hat{F}^{1,1}(\varepsilon[\varepsilon^{-1}x] - \varepsilon\nu_\varepsilon N_\varepsilon, \theta) \qquad (3.20)$$

It is convenient to define

$$\nu_\theta = \pm[\omega'(\theta)\varepsilon^{-1}tN_\varepsilon^{-1}]$$

where the signs $\pm$ are chosen to coincide with those of $\omega(\theta - \varphi)$ in (3.20).

We rewrite (3.19) in the form

$$\frac{1}{16\pi^2} \int_{-\pi}^{\pi} d\theta \, e^{i[l\theta \pm \omega(\theta)\varepsilon^{-1}t]} \int_{-\pi}^{\pi} d\varphi \left(\sum_{\substack{\nu \in J_\varepsilon: \\ \nu < \nu_\theta - 2}} + \sum_{\nu = \nu_\theta - 2}^{\nu_\theta + 2} + \sum_{\substack{\nu \in J_\varepsilon: \\ \nu > \nu_\theta + 2}} \right)$$
$$\cdot \frac{e^{i(\nu + 1/2)N_\varepsilon \varphi} - e^{i(\nu - 1/2)N_\varepsilon \varphi}}{e^{i\varphi} - 1} e^{\pm i\omega(\theta - \varphi)\varepsilon^{-1}t} \hat{F}^{1,1}(\varepsilon[\varepsilon^{-1}x] - \varepsilon\nu N_\varepsilon, \theta)$$
$$(3.21)$$

Now defining the function

$$f(\varphi) = \begin{cases} (e^{i\varphi} - 1)^{-1} i\varphi, & \varphi \neq 0 \\ 1, & \varphi = 0 \end{cases}$$

and substituting variables

$$\nu = \nu_\theta + h$$

we obtain

$$(3.21) = \frac{1}{16\pi^2} \int_{-\pi}^{\pi} d\theta\, e^{i[l\theta \pm \omega(\theta)\varepsilon^{-1}t]} \int_{-\pi}^{\pi} d\varphi\, f(\varphi)$$

$$\cdot\, e^{i[v_\theta N_\varepsilon \varphi \pm \omega(\theta-\varphi)\varepsilon^{-1}t]} \left(\sum_{\substack{-2<h:\\ v_\theta + h \in J_\varepsilon}} + \sum_{h=-2}^{2} + \sum_{\substack{h>2:\\ v_\theta + h \in J_\varepsilon}} \right)$$

$$\cdot\, \frac{e^{i(h+1/2)N_\varepsilon\varphi} - e^{i(h-1/2)N_\varepsilon\varphi}}{i\varphi} \hat{F}^{1,1}(\varepsilon[\varepsilon^{-1}x] - \varepsilon v_\theta N_\varepsilon - \varepsilon h N_\varepsilon, \theta) \qquad (3.22)$$

The right-hand side of (3.22) is written as the sum of three terms corresponding to the sums in parentheses.

The next statement may be considered the principal step in the proof of Theorem 3.1.

Proposition 3.6. The following relation holds

$$\lim_{\varepsilon \to 0} (3.22)_1 = \lim_{\varepsilon \to 0} (3.22)_3 = 0$$

Proof of Proposition 3.6. We consider in detail the sum $(3.22)_1$; the other is treated in the same way. It is convenient to introduce the set $K_\theta^- = K_{\theta,\varepsilon}^- = (J_\varepsilon - v_0) \cap (-\infty, -3] \cap \mathbb{Z}^1$. In order to perform the calculation we rewrite the sum in $(3.22)_1$ in the following form

$$\frac{1}{16\pi^2} \sum_{h \in K_\theta^-} \frac{1}{i\varphi} (e^{i(h+1/2)N_\varepsilon\varphi} - e^{-i(h_{(-)}-1/2)N_\varepsilon\varphi}$$
$$+ e^{-i(h_{(-)}-1/2)N_\varepsilon\varphi} - e^{i(h-1/2)N_\varepsilon\varphi})\, \hat{F}^{1,1}(\varepsilon[\varepsilon^{-1}x] - \varepsilon v_\theta N_\varepsilon - \varepsilon h N_\varepsilon, \theta) \qquad (3.23)$$

where

$$h_{(-)} = \min[k: h \in K_\theta^-]$$

Using a "discrete integration-by-parts formula" we find that (3.23) equals

$$\frac{1}{16\pi^2} \cdot \frac{1}{i\varphi} \left\{ (e^{-i(5/2)N_\varepsilon\varphi} - e^{-i(h_{(-)}-1/2)N_\varepsilon\varphi})\, \hat{F}^{1,1}(\varepsilon[\varepsilon^{-1}x] - \varepsilon v_\theta N_\varepsilon + 3\varepsilon N_\varepsilon, \theta) \right.$$
$$+ \sum_{h \in K_\theta^- : h < -3} (e^{i(h+1/2)N_\varepsilon\varphi} - e^{-i(h_{(-)}-1/2)N_\varepsilon\varphi})$$
$$\cdot \{ \hat{F}^{1,1}(\varepsilon[\varepsilon^{-1}x] - \varepsilon v_\theta N_\varepsilon - \varepsilon h N_\varepsilon, \theta)$$
$$\left. - \hat{F}^{1,1}(\varepsilon[\varepsilon^{-1}x] - \varepsilon v_\theta N_\varepsilon - \varepsilon (h+1) N_\varepsilon, \theta) \right\} \qquad (3.24)$$

Hence, we can write

$$|(3.22)_1| \leqslant \frac{1}{16\pi^2} \int_{-\pi}^{\pi} d\theta \sup_{h \in K_\theta^-} \left| \int_{-\pi}^{\pi} d\varphi\, f(\varphi)\, e^{i(v_\theta N_\varepsilon \varphi \pm \omega(\theta - \varphi)\varepsilon^{-1} t)} \cdot \frac{1}{i\varphi} \left(e^{i(h - 1/2) N_\varepsilon \varphi} - e^{-i(h_{(-)} - 1/2) N_\varepsilon \varphi}\right) \right| \cdot \left\{ \sum_{h \in K_\theta^-} |\hat{F}^{1,1}(\varepsilon[\varepsilon^{-1} x] - \varepsilon v_\theta N_\varepsilon - \varepsilon h N_\varepsilon, \theta) - \hat{F}^{1,1}(\varepsilon[\varepsilon^{-1} x] - \varepsilon v_\theta N_\varepsilon - \varepsilon(h+1) N_\varepsilon, \theta)| + |\hat{F}^{1,1}(\varepsilon[\varepsilon^{-1} x] - \varepsilon v_\theta N_\varepsilon + 3\varepsilon N_\varepsilon, \theta)| \right\} \tag{3.25}$$

The sum over h in (3.25) does not exceed the variation of $\hat{F}^{1,1}(\cdot, \theta)$ on the interval $(x - 4 - ct,\ x + 4 + ct)$. This is bounded uniformly in θ due to condition (D) imposed on $\hat{F}$ (see Section 3). Similary, the last term in parentheses is bounded uniformly in θ. Hence we need to show that the supremum in h in (3.25) tends to zero as $\varepsilon \to 0$ for any $\theta \in [-\pi, \pi)$. For this we apply the following assertion (which will be repeatedly used below).

Lemma 3.7. Under the stated hypotheses on the function ω uniformly in $\theta \in [-\pi, \pi)$ and $k, k' \in \mathbb{R}^1$ with $|k|, |k'| > 2$, we have

$$\lim_{\varepsilon \to 0} \left[\int_{-\pi}^{\pi} d\varphi\, f(\varphi)\, e^{i(v_\theta N_\varepsilon \varphi \pm \omega(\theta + \varphi)\varepsilon^{-1} t)} \frac{1}{i\varphi} \left(e^{ikN_\varepsilon \varphi} - e^{ik' N_\varepsilon \varphi}\right) - \pi e^{\pm i\omega(\theta)\varepsilon^{-1} t} (\operatorname{sgn} k - \operatorname{sgn} k') \right] = 0 \tag{3.26}$$

(The proof of this lemma will be deferred a few lines.)

Thus, we conclude that the right-hand-side of (3.25) approaches zero when $\varepsilon \to 0$. We now move on to an investigation of the second term on the right-hand side of (3.22). We need another technical lemma which, along with Lemma 3.7, is proved below.

Lemma 3.8. Under the stated hypotheses on ω and $\hat{F}$ we have

$$(3.22)_2 = \frac{1}{16\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} d\theta\, e^{i(l\theta \pm \omega(\theta)\varepsilon^{-1} t)}\, \hat{F}^{1,1}[x \mp \omega'(\theta) t, \theta] \int_{-\pi}^{\pi} d\varphi\, f(\varphi) \cdot e^{iv_\theta N_\varepsilon \varphi} \frac{1}{i\varphi} \left(e^{i(5/2) N_\varepsilon \varphi} - e^{-i(5/2) N_\varepsilon \varphi}\right) e^{\pm i\omega(\theta - \varphi)\varepsilon^{-1} t} + G_5(\varepsilon, t) \tag{3.27}$$

where the sign in $\hat{F}^{1,1}[x \mp \omega'(\theta)t, \theta]$ is chosen to be opposite that which appears in front of $\omega(\theta-\varphi)$ and $\lim_{\varepsilon\to 0} G_5(\varepsilon, t)=0$ for any $t\in\mathbb{R}^1$.

Using Lemma 3.8 and Lemma 3.7 we finish the proof of Theorem 3.1. In fact

$$\lim_{\varepsilon\to 0}(3.27)_1 = \frac{1}{8\pi}\lim_{\varepsilon\to 0}\int_{-\pi}^{\pi} d\theta\, e^{i[l\theta \pm \omega(\theta)\varepsilon^{-1}t]}\, \hat{F}^{1,1}(x \mp \omega'(\theta)t, \theta)\, e^{\pm i\omega(\theta)\varepsilon^{-1}t}$$

The nonzero contribution is given in the cases of different signs in exponents. In these cases one arrives at the expression

$$\left.\frac{1}{8\pi}\int_{-\pi}^{\pi} d\theta\, e^{il\theta}\, \hat{F}^{1,1}[x \mp \omega'(\theta)t, \theta\right] \tag{3.28}$$

Putting these results in (3.19) we find that the limit of $(3.15)_{1,1}$ is equal to

$$\frac{1}{8\pi}\int_{-\pi}^{\pi} d\theta\, e^{il\theta}\{\hat{F}^{1,1}[x-\omega'(\theta)t, \theta] + \hat{F}^{1,1}[x+\omega'(\theta)t, \theta]\}$$

which is nothing but the inverse Fourier transform of $(1.7\mathrm{a})_1$.

In the same way one analyzes the other terms on the right-hand side of (3.15), which lead to other addends on the right-hand side of (1.7a).

Now we must prove Lemma 3.7 and Lemma 3.8.

Proof of Lemma 3.7. It is convenient to rewrite (3.26) in the form

$$\lim_{\varepsilon\to 0} e^{\mp i\omega(\theta)\varepsilon^{-1}t}\int_{-\pi}^{\pi} d\varphi\, f(\varphi)\, e^{i[v_\theta N_\varepsilon\varphi \pm \omega(\theta-\varphi)\varepsilon^{-1}t]}$$
$$\cdot\frac{1}{i\varphi}(e^{ikN_\varepsilon\varphi} - e^{ik'N_\varepsilon\varphi}) = \pi(\operatorname{sgn} k - \operatorname{sgn} k')$$

Let's start with the following relations

$$v_\theta N_\varepsilon = \omega'(\theta)\varepsilon^{-1}t + c(\theta, t, \varepsilon)N_\varepsilon \tag{3.28}$$

where $|c(\theta, t, \varepsilon)| < 1$ and

$$\omega(\theta-\varphi) = \omega(\theta) - \omega'(\theta)\varphi + h(\theta, \varphi)\varphi^2$$

where $h(\theta, \varphi)$ is an analytic function such that $h(\theta, 0) = \frac{1}{2}\omega''(\theta)$. It turns out that we must evaluate an integral of the following kind

$$\int_{-\pi}^{\pi} d\varphi\, e^{\pm ih(\theta,\varphi)\varphi^2\varepsilon^{-1}t}\,(f(\varphi)/\varphi)(e^{i[k+c(\theta,t,\varepsilon)]N_\varepsilon\varphi} - e^{i[k'+c(\theta,t,\varepsilon)]N_\varepsilon\varphi})$$

Fix a number $a \in (0, \pi]$ and rewrite the integral under consideration as the sum of the three integrals related to the intervals $[-\pi, -\varepsilon^{1/2} a)$, $[-\varepsilon^{1/2}a, \varepsilon^{1/2}a]$, $(\varepsilon^{1/2}a, \pi]$. We denote these integrals I_1, I_2, I_3, respectively. As will be shown below, I_1 and I_3 give the zero contribution when $\varepsilon \to 0$. For evaluating I_2 we perform the changement of variables $\varphi = \varepsilon^{1/2} z$. We have the following equality

$$I_2 = I_2' + I_2''$$

where

$$I_2' = \int_{-a}^{a} dz (e^{\pm ih(\theta, \varepsilon^{1/2} z) z^2 t} - e^{\pm ih(\theta, 0) z^2 t}) \frac{f(\varepsilon^{1/2} z)}{z} \cdot (e^{i[k + c(\theta, t, \varepsilon)] N_\varepsilon \varepsilon^{1/2} z} - e^{i[k' + c(\theta, t, \varepsilon)] N_\varepsilon \varepsilon^{1/2} z})$$

$$I_2'' = \int_{-a}^{a} dz\, e^{\pm ih(\theta, 0) z^2 t} \frac{f(\varepsilon^{1/2} z)}{z} (e^{i[k + c(\theta, t, \varepsilon)] N_\varepsilon \varepsilon^{1/2} z} - e^{i[k' + c(\theta, t, \varepsilon)] N_\varepsilon \varepsilon^{1/2} z})$$

Obviously

$$|I_2'| \leqslant \text{const} \int_{-a}^{a} |h(\theta, \varepsilon^{1/2} z) - h(\theta, 0)|\, |z|\, dz$$

and the right-hand side vanishes as $\varepsilon \to 0$ uniformly in θ and in k, k'.

Now

$$\begin{aligned} I_2'' = {} & \varepsilon^{1/2} \int_{-a}^{a} dz\, e^{\pm ih(\theta, 0) z^2 t} \frac{1}{\varepsilon^{1/2} z} [f(\varepsilon^{1/2} z) - f(0)] \\ & \cdot (e^{i[k + c(\theta, t, \varepsilon)] N_\varepsilon \varepsilon^{1/2} z} - e^{i[k' + c(\theta, t, \varepsilon)] N_\varepsilon \varepsilon^{1/2} z}) \\ & + f(0) \int_{-a}^{a} dz\, e^{\pm ih(\theta, 0) z^2 t} \frac{1}{z} (e^{i[k + c(\theta, t, \varepsilon)] N_\varepsilon \varepsilon^{1/2} z} \\ & - e^{i[k' + c(\theta, t, \varepsilon)] N_\varepsilon \varepsilon^{1/2} z}) \end{aligned} \tag{3.29}$$

The first term on the right-hand side of (3.29) is vanishing as $\varepsilon \to 0$ uniformly in θ and k, k'. In order to evaluate the second one we must use the previous trick again

$$\begin{aligned} (3.29)_2 = {} & \int_{-a}^{a} dz \frac{1}{z} (e^{\pm ih(\theta, 0) z^2 t} - 1)(e^{i[k + c(\theta, t, \varepsilon)] N_\varepsilon \varepsilon^{1/2} z} \\ & - e^{i[k' + c(\theta, t, \varepsilon)] N_\varepsilon \varepsilon^{1/2} z}) + \int_{-a}^{a} dz \frac{1}{z} (e^{i[k + c(\theta, t, \varepsilon)] N_\varepsilon \varepsilon^{1/2} z} \\ & - e^{i[k' + c(\theta, t, \varepsilon)] N_\varepsilon \varepsilon^{1/2} z}) \end{aligned} \tag{3.30}$$

The first term on the right-hand side of (3.30) vanishes as $\varepsilon \to 0$ via the integration by parts uniformly in θ and k, k'. So we have to demonstrate that uniformly in θ and k, k'

$$\lim_{\varepsilon \to 0} (3.30)_2 = \pi i(\operatorname{sgn} k - \operatorname{sgn} k')$$

This is a straightforward calculation. Let $\Gamma_a^{\pm}$ denote the upper $(+)$ and lower $(-)$ semicircumference of radius a centered at the origin and oriented from a to $-a$. Due to the analyticity of the integrand in $(3.30)_2$

$$(3.30)_2 = \int_{\Gamma_a^{\pm}} dz \frac{1}{z} e^{i[k + c(\theta,t,\varepsilon)] N_\varepsilon \varepsilon^{1/2} z} - \int_{\Gamma_a^{\pm}} dz \frac{1}{z} e^{i[k' + c(\theta,t,\varepsilon)] N_\varepsilon \varepsilon^{1/2} z} \tag{3.31}$$

Since $|k|$, $|k'| > 2$ and $|c(\theta, t, \varepsilon)| < 1$

$$\operatorname{sgn} k = \operatorname{sgn}[k + c(\theta, t, \varepsilon)], \qquad \operatorname{sgn} k' = \operatorname{sgn}[k' + c(\theta, t, \varepsilon)]$$

Let us assume first that $\operatorname{sgn}[k + c(\theta, t, \varepsilon)] = \operatorname{sgn}[k' + c(\theta, t, \varepsilon)]$. We choose, in the right-hand side of (3.31), Γ_a^+ if sgn $= +1$ and Γ_a^- if sgn $= -1$. Using integration by parts it is easy to see that in both cases both integrals do not exceed const $N_\varepsilon^{-1} \varepsilon^{-1/2}$ and hence, vanish as $\varepsilon \to 0$ (see 3.5). Moreover, the constant is bounded uniformly in θ and k, k'.

Finally, if $\operatorname{sgn}[k + c(\theta, t, \varepsilon)] \neq \operatorname{sgn}[k' + c(\theta, t, \varepsilon)]$, we use either the formula

$$\int_{\Gamma_a^-} dz \frac{1}{z} e^{i[k + c(\theta,t,\varepsilon)] N_\varepsilon \varepsilon^{1/2} z} = \int_{\Gamma_a^+} dz \frac{1}{z} e^{i[k + c(\theta,t,\varepsilon)] N_\varepsilon \varepsilon^{1/2} z} + 2\pi i$$

or a similar formula replacing k by k'. The result follows immediately.

We now move on to the estimation of I_1 and I_3. For the sake of brevity, consider in detail the integral I_1; I_3 has essentially the same behavior. Observe that it is enough to show that

$$\lim_{\varepsilon \to 0} \int_{-\pi}^{-a\varepsilon^{1/2}} d\varphi \frac{f(\varphi)}{\varphi} e^{i[\pm h(\theta,\varphi)\varphi^2 \varepsilon^{-1} t + kN_\varepsilon \varphi]} = 0 \tag{3.32}$$

uniformly in k with $|k| > 1$ and $\theta \in [-\pi, \pi)$.

Integrating by parts we obtain that the integral in (3.32) is equal to

$$\begin{aligned} &\frac{1}{ikN_\varepsilon} \left[\frac{f(\varphi)}{\varphi} e^{i[\pm h(\theta,\varphi)\varphi^2 \varepsilon^{-1} t + kN_\varepsilon \varphi]} \right]_{-\pi}^{-a\varepsilon^{1/2}} \\ &\qquad - \frac{1}{ikN_\varepsilon} \int_{-\pi}^{-a\varepsilon^{1/2}} d\varphi\, e^{ikN_\varepsilon \varphi} \frac{d}{d\varphi} \left(e^{\pm ih(\theta,\varphi)\varphi^2 \varepsilon^{-1} t} \frac{f(\varphi)}{\varphi} \right) \end{aligned} \tag{3.33}$$

The term $(3.33)_1$ vanishes via (3.5) uniformly in k and θ. The second term on the right-hand side of (3.33) becomes

$$\frac{1}{ikN_\varepsilon \varepsilon}\int_{-\pi}^{-a\varepsilon^{1/2}} d\varphi\, e^{i[kN_\varepsilon\varphi \pm h(\theta,\varphi)\varphi^2\varepsilon^{-1}t]} f(\varphi)[\pm ih'_\varphi(\theta,\varphi)\,\varphi t \pm 2ih(\theta,\varphi)t]$$
$$+\frac{1}{ikN_\varepsilon}\int_{-\pi}^{-a\varepsilon^{1/2}} d\varphi\,\frac{f'(\varphi)}{\varphi}\, e^{i[kN_\varepsilon\varphi \pm h(\theta,\varphi)\varphi^2\varepsilon^{-1}t]}$$
$$-\frac{1}{ikN_\varepsilon}\int_{-\pi}^{-a\varepsilon^{1/2}} d\varphi\,\frac{f(\varphi)}{\varphi^2}\, e^{i[kN_\varepsilon\varphi \pm h(\theta,\varphi)\varphi^2\varepsilon^{-1}t]} \tag{3.34}$$

To estimate the first term on the right-hand side of (3.34) we notice that, according to the definition of the function h and condition (iv) on the function ω (see text following Definition 2.3), the set

$$\left\{\varphi: \frac{d^2}{d\varphi^2}[h(\theta,\varphi)\varphi^2] = \frac{d^3}{d\varphi^3}[h(\theta,\varphi)\varphi^2] = 0\right\}$$

is empty. Hence, due to Lemma 2.6(ii), the term $(3.34)_1$ is less than or equal to const. $(1/|k|\,N_\varepsilon\varepsilon)\cdot\varepsilon^{1/3}$ which vanishes as $\varepsilon\to 0$ in view of (3.5). The constant is uniformly bounded in θ.

The remaining terms on the right-hand side of (3.34) are estimated by using a straightforward bound of the integrand and do not exceed, respectively, const $|\ln\varepsilon|/N_\varepsilon$ and const. $1/\varepsilon^{1/2}N_\varepsilon$. Both bounds vanish as $\varepsilon\to 0$ due to (3.5) and both constants are uniformly bounded in θ. This completes the proof.

Proof of Lemma 3.8. By using the inequality

$$|(\varepsilon[\varepsilon^{-1}x] - \varepsilon v_\theta N_\varepsilon - \varepsilon h N_\varepsilon) - [x \mp \omega'(\theta)t]| \leqslant \varepsilon(1+5N_\varepsilon)$$

which follows from the definitions of v_θ (see (3.28)) and from the bound $|h|\leqslant\frac{5}{2}$ and the smoothness properties of $\hat{F}^{\gamma,\gamma'}(x,\cdot)$ (see condition D). We conclude that

$$|G_5(\varepsilon,t)| \leqslant \text{const}\,\varepsilon(1+5N_\varepsilon)\sum_{h=-2}^{2}\int_{-\pi}^{\pi} d\theta\left|\int_{-\pi}^{\pi} d\varphi\,\frac{f(\varphi)}{i\varphi}\right.$$
$$\left.\cdot\, e^{i[v_\theta N_\varepsilon\varphi \pm\omega(\theta-\varphi)\varepsilon^{-1}t]}(e^{i(h+1/2)N_\varepsilon\varphi} - e^{i(h-1/2)N_\varepsilon\varphi})\right| \tag{3.35}$$

The problem is reduced to estimating the integral $\int d\varphi$ in a single term on the right-hand side of (3.35). We rewrite it as the sum of three integrals related to the intervals $[-\pi, -\varepsilon)$, $[-\varepsilon, \varepsilon]$, and $(\varepsilon, \pi]$. We again denote these integrals I_1, I_2, I_3, respectively. The integrals I_1, I_3 do not exceed

$$2\int_{\varepsilon}^{\pi} d\varphi \; |f(\varphi)|/\varphi \leqslant \text{const } |\ln \varepsilon|$$

and the integral I_2 is bounded by

$$\int_{-\varepsilon}^{\varepsilon} d\varphi \; |f(\varphi)/\varphi| \; |e^{i(h+1/2)N_\varepsilon \varphi} - e^{i(h-1/2)N_\varepsilon \varphi}| \leqslant \text{const} \cdot \varepsilon N_\varepsilon$$

Substituting this into (3.35) gives the result in view of (3.5).

4. LOCALLY CONSERVED QUANTITIES

We have seen from Proposition 2.4 that time-invariant Gaussian states may be characterized by means of the quantities E and A (see 2.30, 2.31) which generate the first integrals e and a (see 2.34, 2.35). We study in this section the "locally conserved" quantities X_k^ε and Y_k^ε which are introduced in the following:

Definition 4.1. Let φ: $\mathbb{R}^1 \to \mathbb{R}^1$ be a C_0^1 function. We set

$$X_k^\varepsilon(\varphi; \mathbf{y}) = \varepsilon \sum_{h \in \mathbb{Z}^1} \varphi(\varepsilon h) \, E(h, h+k, \mathbf{y}) \tag{4.1}$$

$$Y_k^\varepsilon(\varphi; \mathbf{y}) = \varepsilon \sum_{h \in \mathbb{Z}^1} \varphi(\varepsilon h) \, A(h, h+k, \mathbf{y}) \tag{4.2}$$

Theorem 4.1. Let a matrix function $\hat{F}$ be given for which conditions (A)–(D) of Section 3 hold. Let $\{P^\varepsilon, \varepsilon > 0\}$ be a hydrodynamic family of states for $\hat{F}$. Then, for any $\varphi \in C_0^1(\mathbb{R}^1)$, $k \in \mathbb{Z}^1$, and nonzero $t \in \mathbb{R}^1$ there exist the limits

$$\lim_{\varepsilon \to 0} \langle X_k^\varepsilon(\varphi; \cdot) \rangle_{P^\varepsilon_{\varepsilon^{-1}t}} = E_k(\varphi; t) \tag{4.3}$$

$$\lim_{\varepsilon \to 0} \langle Y_k^\varepsilon(\varphi; \cdot) \rangle_{P^\varepsilon_{\varepsilon^{-1}t}} = A_k(\varphi; t) \tag{4.4}$$

The Fourier transforms

$$\hat{E}(\varphi; t, \theta) = \sum_{k \in \mathbb{Z}^1} E_k(\varphi; t) e^{ik\theta}, \qquad \hat{A}(\varphi; t, \theta) = \sum_{k \in \mathbb{Z}^1} A_k(\varphi; t) e^{ik\theta}$$

are given by

$$\hat{E}(\varphi; t, \theta) = \frac{1}{2}\int dx\, \varphi(x)[\omega(\theta)^2\, \hat{F}^{1,1}(t; x, \theta) + \hat{F}^{2,2}(t; x, \theta)] \tag{4.5}$$

$$\hat{A}(\varphi; t. \theta) = \frac{1}{2}\int dx\, \varphi(x)[\hat{F}^{1,2}(t; x, \theta) - \hat{F}^{2,1}(t; x, \theta)] \tag{4.6}$$

where $\hat{F}^{\gamma,\gamma'}(t; x, \theta)$, $\gamma, \gamma' = 1, 2$, are defined in (3.10a–d).

The proof of Theorem 4.1 is based on the Lebesgue-dominated convergence theorem and Theorem 3.1. The key remark is that the expectations

$$\langle E(h, h+k; \cdot)\rangle_{P^{\varepsilon}_{\varepsilon^{-1}t}}, \qquad \langle A(h, h+k; \cdot)\rangle_{P^{\varepsilon}_{\varepsilon^{-1}t}}$$

are bounded uniformly in $\varepsilon > 0$ and $h, k \in \mathbb{Z}^1$ (see Lemma 2.7).

Remarks (1). The limit values $E_k(\varphi; t)$ and $A_k(\varphi; t)$ coincide with the integrals

$$\int dx\, \varphi(x)\langle E(0, k; \cdot)\rangle_{G_{t,x}} \tag{4.7}$$

and

$$\int dx\, \varphi(x)\langle A(0, k; \cdot)\rangle_{G_{t,x}} \tag{4.8}$$

respectively. Here $G_{t,x}$ denotes the Gaussian state with the SDMF $\hat{F}(t; x, \cdot)$.

(2) From (4.5) and (4.6), together with (1.7a–d), it follows that $\hat{E}$ and $\hat{A}$ satisfy the equations

$$\frac{\partial}{\partial t}\hat{E}(\varphi; t, \theta) = -i\omega'(\theta)\,\omega(\theta)\,\hat{A}(\varphi'; t, \theta) \tag{4.9}$$

$$\frac{\partial}{\partial t}\hat{A}(\varphi; t, \theta) = i\frac{\omega'(\theta)}{\omega(\theta)}\hat{E}(\varphi'; t, \theta) \tag{4.10}$$

(cf. 1.4 and 1.5).

Our aim in this section is to prove a stronger result:

Theorem 4.2. Let the conditions of Theorem 4.1 be fulfilled. Assume in addtion that every state $P^{\varepsilon} \in \mathfrak{B}$ (see Lemma 2.7) and

$\sup_{\varepsilon>0} \sup_{j\in\mathbb{Z}^1}\langle \| y_j\|^5\rangle_{P^\varepsilon} < \infty$. Then, for any $\eta > 0$, $\varphi\in C_0^1(\mathbb{R}^1)$, $k\in\mathbb{Z}^1$ and nonzero $t\in\mathbb{R}^1$

$$\lim_{\varepsilon\to 0} P^\varepsilon_{\varepsilon^{-1}t}\{|X^\varepsilon_k(\varphi;\cdot)-E_k(\varphi;t)|<\eta\}=1 \tag{4.11}$$

$$\lim_{\varepsilon\to 0} P^\varepsilon_{\varepsilon^{-1}t}\{|Y^\varepsilon_k(\varphi;\cdot)-A_k(\varphi;t)|<\eta\}=1 \tag{4.12}$$

Before giving the proof of Theorem 4.2 we formulate an assertion (Theorem 4.3) which gives a somewhat stronger result than Theorem 3.1.

Theorem 4.3. Let the conditions of Theorem 4.2 be fulfilled. Then, for any $x\in\mathbb{R}^1$ and nonzero $t\in\mathbb{R}^1$, the states $S_{-[\varepsilon^{-1}x]}P^\varepsilon_{\varepsilon^{-1}t}$, $\varepsilon>0$ converge, as $\varepsilon\to 0$, in the vague topology, to the Gaussian state $G_{t,x}$. Moreover, for every bounded I_1, $I_2 \dot{\subset} \mathbb{Z}^1$ the distributions and expectations of the absolute values of the centered sums

$$\begin{gathered}\left|\sum_{h\in I_1}\sum_{k\in I_2}(E(h,h+k;\cdot)-\langle E(h,h+k;\cdot)\rangle_{S_{-[\varepsilon^{-1}x]}P^\varepsilon_{\varepsilon^{-1}tx}}\right| \\ \left|\sum_{h\in I_1}\sum_{k\in I_2}(A(h,h+k;\cdot)-\langle A(h,h+k;\cdot)\rangle_{S_{-[\varepsilon^{-1}x]}P^\varepsilon_{\varepsilon^{-1}t}}\right|\end{gathered} \tag{4.13}$$

w.r.t. states $S_{-[\varepsilon^{-1}x]}P^\varepsilon_{\varepsilon^{-1}t}$ converge, as $\varepsilon\to 0$, to the distributions and expectations of the absolute values of the limiting centered sums

$$\begin{gathered}\left|\sum_{h\in I_1}\sum_{k\in I_2}(E(h,h+k;\cdot)-\langle E(h,h+k;\cdot)\rangle_{G_{t,x}})\right| \\ \left|\sum_{h\in I_1}\sum_{k\in I_2}(A(h+k;\cdot)-\langle A(h,h+k;\cdot)\rangle_{G_{t,x}})\right|\end{gathered} \tag{4.13$'$}$$

w.r.t. the Gaussian states $G_{t,x}$. This convergence is uniform in x within any fixed bounded interval of $\mathbb{R}^1$.

The proof of Theorem 4.3 is based on a combination of arguments used in Ref. 8 and in Section 3 of this paper. The most delicate point is to prove the convergence of the random variable (4.13) to (4.13′). Here one uses the assertion of Lemma 2.7. (For the sake of brevity we do not supply the details.)

Proof of Theorem 4.2. For the sake of brevity we discuss the relation (4.11): similar arguments apply to proving (4.12). To prove (4.11) it suffices to demonstrate that

$$\lim_{\varepsilon\to 0}\langle |X^\varepsilon_k(\varphi;\cdot)-E_k(\varphi;t)|\rangle_{P^\varepsilon_{\varepsilon^{-1}t}}=0$$

or, in view of Theorem 4.1

$$\lim_{\varepsilon \to 0} \langle |X_k^\varepsilon(\varphi; \cdot) - \langle X_k^\varepsilon(\varphi; \cdot)\rangle_{P^\varepsilon_{\varepsilon^{-1}t}}| \rangle_{P^\varepsilon_{\varepsilon^{-1}t}} = 0 \tag{4.14}$$

Let L be a positive integer to be chosen later. We rewrite (4.1) in the form

$$X_k^\varepsilon(\varphi; \mathbf{y}) = \varepsilon L \sum_{h \in \mathbb{Z}^1} \frac{1}{L} \sum_{h=nL}^{(n+1)L-1} \varphi(\varepsilon h)\, E(h, h+k; \mathbf{y}) \tag{4.15}$$

and estimate

$$\begin{aligned} &\langle |X_k^\varepsilon(\varphi; \cdot) - \langle X_k^\varepsilon(\varphi; \cdot)\rangle_{P^\varepsilon_{\varepsilon^{-1}t}}| \rangle_{P^\varepsilon_{\varepsilon^{-1}t}} \\ &\quad \leqslant \varepsilon L \sum_{n \in \mathbb{Z}^1} \left\langle \left| \frac{1}{L} \sum_{h=nL}^{(n+1)L-1} \varphi(\varepsilon h)[E(h, h+k; \cdot) \right.\right. \\ &\quad\quad \left.\left. - \langle E(h, h+k; \cdot)\rangle_{P^\varepsilon_{\varepsilon^{-1}t}}] \right| \right\rangle_{P^\varepsilon_{\varepsilon^{-1}t}} \end{aligned} \tag{4.16}$$

Furthermore, in view of the smoothness properties of φ and the uniform boundedness of $\langle |E(h, h+k; \cdot)| \rangle_{P^\varepsilon_{\varepsilon^{-1}t}}$, $\varepsilon > 0$, h, $k \in \mathbb{Z}^1$ (see Lemma 2.7)

$$\begin{aligned} &\text{r.h.s. of } (4.16) \leqslant \varepsilon L \sum_{h \in \mathbb{Z}^1} |\varphi(\varepsilon h L)| \\ &\quad \cdot \left\langle \left| \frac{1}{L} \sum_{h=nL}^{(n+1)L-1} (E(h, h+k; \cdot) - \langle E(h, h+k; \cdot)\rangle_{P^\varepsilon_{\varepsilon^{-1}t}}) \right| \right\rangle_{P^\varepsilon_{\varepsilon^{-1}t}} + L \cdot O(\varepsilon) \end{aligned} \tag{4.17}$$

Now, according to Theorem 4.3, for any fixed x, t and L

$$\begin{aligned} &\lim_{\varepsilon \to 0} \left\langle \left| \frac{1}{L} \sum_{h=[\varepsilon^{-1}x]}^{[\varepsilon^{-1}x]+L-1} (E(h, h+k; \cdot) - \langle E(h, h+k; \cdot)\rangle_{P^\varepsilon_{\varepsilon^{-1}t}}) \right| \right\rangle_{P^\varepsilon_{\varepsilon^{-1}t}} \\ &\quad = \left\langle \left| \frac{1}{L} \sum_{h=0}^{L-1} (E(h, h+k; \cdot) - \langle E(h, h+k; \cdot)\rangle_{G_{t,x}}) \right| \right\rangle_{G_{t,x}} \end{aligned} \tag{4.18}$$

and the convergence is uniform in $x \in \operatorname{supp} \varphi$. Hence, for any $L \geqslant 1$

$$\limsup_{\varepsilon \to 0}\, (\text{r.h.s. of } 4.16) \leqslant \int dx\, |\varphi(x)| \cdot (\text{r.h.s. of } 4.18) \tag{4.19}$$

Now we have to prove that the right-hand side of (4.19) vanishes as $L \to \infty$. In view of the Lebesgue-dominated convergence theorem, it is sufficient to check that the right-hand side of (4.18) vanishes as $L \to \infty$ for any fixed $x \in \mathbb{R}^1$ and nonzero $t \in \mathbb{R}^1$, and that it is bounded uniformly in $x \in \operatorname{supp} \varphi$. The latter assertion follows from the estimates

$$\begin{aligned}\text{r.h.s. of (4.18)} &\leqslant \left\langle \frac{1}{L} \sum_{h=0}^{L-1} |E(h, h+k; \cdot)| \right\rangle_{G_{t,x}} + |\langle E(h, h+k; \cdot) \rangle_{G_{t,x}}| \\ &\leqslant 2 \langle |E(h, h+k; \cdot)| \rangle_{G_{t,x}} \leqslant \text{const} \langle \| y_j \|^2 \rangle_{G_{t,x}} \\ &= \text{const} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta [\hat{F}^{1,1}(t; x, \theta) + \hat{F}^{2,2}(t; x, \theta)]\end{aligned}$$

and from formulas (1.7a,d) and condition (D) on the initial matrix function $\hat{F}(x, \theta)$ (see Section 3).

The convergence of the right-hand side of (4.18) to zero follows from the ergodicity of the state $G_{t,x}$.

5. FAMILIES OF INITIAL STATES SATISFYING THE CONDITIONS OF THEOREMS 3.1 AND 4.2

In this section we discuss examples of families $\{P^\varepsilon, \varepsilon > 0\}$ satisfying the conditions of Theorems 3.1 and 4.2. Such examples are provided by the theory of Gibbs states. For the sake of simplicity we impose assumptions on the potentials stronger than those which are necessary for our constructions; this allows us to use results and methods of some previous papers. Possible ways to generalize our statements are briefly mentioned in the end of the section.

Let $\varphi(x, y)$, $x \in \mathbb{R}^1$, $y = (q, p) \in \mathbb{R}^1 \times \mathbb{R}^1$ be a real function of the form

$$\varphi(x, y) = \varphi_1(y) + \varphi_2(x, y)$$

where (A) for any $s > 0$

$$\int dy (1 + \| y \|)^5 \, \varphi_1(y) \exp[-s\varphi_1(y)] < \infty \tag{5.1}$$

and (B) φ_2 is of class C^1 in the variables x and is bounded together with the derivative $(\partial \varphi_2 / \partial x)$ uniformly in $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1 \times \mathbb{R}^1$.

We consider the potentials $\Phi_\varepsilon = \{\Phi_\varepsilon^{(\Lambda)}, \Lambda \dot{\subset} \mathbb{Z}^1\}$ satisfying the following assumptions:

(1) Φ_ε is a potential of finite range: there exists d_0 (which does not depend on ε) such that $\Phi_\varepsilon^{(\Lambda)} \equiv 0$ whenever

$$l(\Lambda) = \max_{j_1, j_2 \in \Lambda} |j_1 - j_2| \geqslant d_0 \tag{5.2}$$

(2) The family $\{\Phi^{(\Lambda)}, \operatorname{card} \Lambda \geqslant 2\}$ is translationally invariant and bounded

$$\sup_{\substack{\Lambda \dot{\subset} \mathbb{Z};\, \operatorname{card} \Lambda \geqslant 2 \\ \mathbf{y}^{(\Lambda)} \in \mathscr{X}(\Lambda)}} |\Phi^{(\Lambda)}(\mathbf{y}^{(\Lambda)})| < \infty \tag{5.3}$$

(3) The "one-particle" part $\{\Phi_\varepsilon^{(j)}, j \in \mathbb{Z}^1\}$ of the potential Φ_ε is given by

$$\Phi_\varepsilon^{(j)}(y) = \varphi(\varepsilon j, y), \qquad y \in \mathbb{R}^1 \times \mathbb{R}^1 \tag{5.4}$$

Notice that for any fixed $x \in \mathbb{R}^1$ we have a translationally invariant potential $\Psi_x = \{\Psi_x^{(\Lambda)}, \Lambda \dot{\subset} \mathbb{Z}^1\}$ defined by

$$\Psi_x^{(\Lambda)}(\mathbf{y}^{(\Lambda)}) = \begin{cases} \varphi(x, y), & \text{if } \operatorname{card} \Lambda = 1,\ \Lambda = \{j\},\ \mathbf{y}^{(\Lambda)} = y \quad (5.5) \\ \Phi^{(\Lambda)}, & \text{if } \operatorname{card} \Lambda \geqslant 2 \quad (5.6) \end{cases}$$

Proposition 5.1. Given $x \in \mathbb{R}^1$ there exists the unique Gibbs state Q_x corresponding to the generating potential Ψ_x. The state Q_x is translationally invariant and has all the moments

$$\langle \| y_j \|^s \rangle_{Q_x} < \infty, \qquad s > 0 \tag{5.7}$$

which are of class C^1, as functions of $x \in \mathbb{R}^1$.

The proof of Proposition 5.1 follows from arguments and methods developed in the series of papers[12] on one-dimensional systems of classical statistical mechanics. The only difference between Proposition 5.1 and theorems from Ref. 12 is that we consider now the system with the non-compact spin space $\mathbb{R}^1 \times \mathbb{R}^1$. However, condition (B) and (1), (2) give a possibility to introduce a "compact function" on $\mathbb{R}^1 \times \mathbb{R}^1$ and thereby to use the compactness argument in a modified from (see, e.g., the series of papers of Ref. 13).

Now we set

$$F_k^{\gamma,\gamma'}(x) = \langle y_0^\gamma y_k^{\gamma'} \rangle_{Q_x}, \qquad \gamma, \gamma' = 1, 2,\ x \in \mathbb{R}^1,\ k \in \mathbb{Z}^1 \tag{5.8}$$

$$\hat{F}^{\gamma,\gamma'}(x, \theta) = \sum_{k \in \mathbb{Z}^1} \hat{F}_k^{\gamma,\gamma'}(x) \exp(ik\theta), \qquad \gamma, \gamma' = 1, 2,\ x \in \mathbb{R}^1,\ \theta \in [-\pi, \pi) \tag{5.9}$$

Theorem 5.2. Let a family of potentials $\{\Phi_\varepsilon, \varepsilon > 0\}$ be given which satisfy conditions (1)–(3). Then for any ε there exists the unique Gibbs state P^ε corresponding to Φ_ε. The family of states $\{P^\varepsilon, \varepsilon > 0\}$ satisfies the conditions of Theorems 3.1 and 4.2 with the SDMF $\hat{F}$ given by (5.8) and (5.9).

The proof of Theorem 5.2 represents a modification of arguments and methods from Ref. 12. Apart from technical details, which are omitted, we give a short sketch of the proof. Assumptions (1) and (2) give the existence, uniqueness, and mixing property of the Gibbs state P^ε while assumption (3) (together with 1 and 2) assures that the state P^ε satisfies condition (a) of Definition 3.1. The principal difference from the arguments from Ref. 12a–d is that we now deal with nontranslationally invariant potentials and state. This is, however, avoided by using "uniform" ergodic (contracting) properties of operators defined by conditional probability distributions of the state P^ε.[12] Condition (1) may be weakened to an assumption of exponential decreasing for the "norm"

$$\sup_{\Lambda \dot{\subset} \mathbb{Z}^1 : \operatorname{card} \Lambda = n} \|\Phi_\varepsilon^{(\Lambda)}\| \qquad \text{as} \quad n \to \infty$$

In condition (2) the assumption that $\Phi^{(\Lambda)}$ depends on ε may be adopted. Then we should reformulate condition (3) not only for the single-spin part $\{\Phi_\varepsilon^{(j)}\}$ but for all $\Phi^{(\Lambda)}$ that are nonzero. Hence, in addition to the function $\varphi(x, y)$, we consider a family of functions $\varphi^{(\Lambda)}(x, \mathrm{y}^{(\Lambda)})$, $\Lambda \dot{\subset} \mathbb{Z}^1$ which, for fixed x, are translationally invariant. The state Q_x will be defined as the Gibbs state corresponding to the generating potential $\varphi(x) = \{\varphi^{(\Lambda)}(x, \cdot), \Lambda \dot{\subset} \mathbb{Z}^1\}$.

We must remark that a very hard analysis is required by dropping the condition that potentials Φ_ε are bounded and of finite range. We will not go into details: the theory of Gibbs states (even for translationally invariant pair potentials) is not yet developed sufficiently. Some results are given in Ref. 14.

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